Ohio's Systemic Initiative
Discovery

Can You Really Get There From Here?

One-Page Overview

By Robert B. Brown, The Ohio State University

Topics:

Problem-solving strategies, Patterns and relationships, Probability

Grade level:

6 - 12

Getting Started:

You are at an airport on the West coast of a long, narrow country. The country extends lengthwise to the East coast, and there are intermediate airports lined up directly along the way. Your goal is to piece together several nonstop flights and end up at the final airport on the East coast tonight. All of the flights go east only, just the way you are going. There are nonstop flights from every airport to every other one farther east. There is no trouble making connections, because every outgoing flight waits for all incoming passengers before taking off. The trouble is that the scheduling is a bit random today, and the chances that any given flight will go is only fifty-fifty. Furthermore, the statuses of all flights are completely independent of each other. What is the probability that there is any way at all of making the trip today?

Ohio Academic Content Standards, 2002

5-7		8-10		11-12	
1. Number, Number Sense and Operations		1. Number, Number Sense and Operations		1. Number, Number Sense and Operations	
2. Measurement		2. Measurement		2. Measurement	
3. Geometry and Spatial Sense		3. Geometry and Spatial Sense		3. Geometry and Spatial Sense	
4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	x
5. Data Analysis and Probability	X	5. Data Analysis and Probability	X	5. Data Analysis and Probability	X
Mathematical Processes Problem Solving Representation		**Mathematical Processes** Problem Solving Representation		**Mathematical Processes** Problem Solving Representation	

NCTM Principles and Standards, 2000

6-8		9-12	
1. Number and Operations		1. Number and Operations	
2. Algebra	X	2. Algebra	X
3. Geometry		3. Geometry	
4. Measurement		4. Measurement	
5. Data Analysis and Probability	X	5. Data Analysis and Probability	X
6. Problem Solving	X	6. Problem Solving	X
7. Reasoning and Proof		7. Reasoning and Proof	
8. Communication		8. Communication	
9. Connections		9. Connections	
10.Representation	X	10.Representation	X

Note: Capital X denotes major emphasis; lower case x denotes minor emphasis.

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Topics: Problem-solving strategies, Patterns and relationships, Probability	
Levels: Grades 6 – 12	*Timing:* One or more hours.
Materials: None	*Prerequisites:* Familiarity with probability

Problem:

Draw some straight lines in the plane. Into how many regions is the plane divided? This activity can be used as the culmination of a study of probability. It utilizes the computational techniques of elementary probability. Perhaps even more importantly, it can serve as a medium for discussing many different problem-solving techniques. Patterns of widely varying subtlety can be discovered, many of which have strong connections with algebra, number theory, and discrete mathematics. There is something in this problem for everyone.

Goals:

- Practice uncovering patterns
- Review probability
- Sifting out irrelevant ideas
- Problem-solving techniques
- Practice summing a finite geometric series
- Recognizing the power of algebra
- Seeing the computational power of using a letter in place of a numerical constant
- Practice making conjectures and testing them
- Review binomial coefficients and Pascal's triangle
- Practice finding all possibilities in a complex situation

Procedure:

1. Break the class up into groups, and pose the following problem to them:

   It is morning. You are at an airport on the west coast of a long, narrow country. The country extends lengthwise to the east coast, and there are intermediate airports lined up directly along the way. Your goal is to piece together several nonstop flights and end up at the final airport on the east coast tonight. All of the flights go east only, just the way you are going. And there are nonstop flights from every airport to every other one farther east. There is no trouble making connections, because every outgoing flight waits for all incoming passengers before taking off. And don't worry about making it by dinnertime—the planes are super fast. The trouble is that the scheduling is a bit random, and the chances that any particular flight will go on a given day is only fifty-fifty. Furthermore, the statuses of all flights are completely independent of each other. What is the probability that you can get all the way to the East coast today?

2. Let the students discuss the problem for about 10 minutes to see if they understand it. Let each group report its concerns. One of the main concerns should be that the number of airports is not mentioned, and the probability of getting to the east coast could depend on the number of intermediate airports. Also, the students will probably see right away that the probability is at least one-half, no matter how many airports there are, because there is a fifty-fifty chance there will be a nonstop flight today directly from the west coast to the east coast.

3. From the comments of the students, piece together the following abstract model of the problem. Let the succession of airports be represented by the whole numbers 0, 1, 2, ... , $n–1$, n, with 0 being the starting point (west coast) and n being the end (east coast). For each pair of whole numbers $i < j$ flip a coin and if it comes up heads draw an arrow from i to j. What is the probability that there is some connected path of arrows from 0 to n?

4. Have the groups of students calculate what the probability would be if $n = 1$, if $n = 2$, if $n = 3$, and so on. This will give the students a chance to think about probability trees for multistage events and to review the rules for calculating compound probabilities.

5. Have the groups report and discuss their methods and difficulties.

6. The students will see right away that as n gets larger, the work of computing the probability grows quickly. Once they realize this, have them discuss their thoughts on streamlining the calculations.

7. Ask them what they think will happen to the probability as n gets larger and larger.

Closure:

This is a very open-ended problem. It has many connections with other parts of mathematics. Discuss with your students what kind of closure might be satisfactory to them in such a situation.

Extensions:

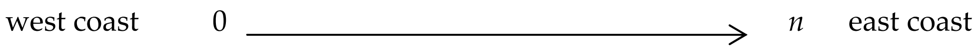
1. Suggest using an undetermined but fixed value p, instead of 1/2, for the probability that there is an arrow from i to j. Let $q = 1 - p$, the probability that there is not an arrow from i to j. Students may see that the use of q (or p) suggests additional patterns that may help navigate the calculations.
2. All of the discussion up to this point has concerned what is the probability that on a given day the traveler can get from coast to coast. However, even if there were a successful routing, whether the traveler will discover it could depend on what information the traveler gets at the airport. If he shows up at the West coast airport and can find out right away which flights are flying that day from *every* airport along the rout, s/he can trace through all the possibilities and choose a successful routing, if there is one.

   But the situation and the probabilities might be different if the traveler can only find out what flights are available from a specific airport by flying to that airport. In that situation when s/he shows up at the West coast airport s/he will know only which airports can be reached in one hop from the west coast. If there is a nonstop flight directly to the east coast, s/he will take it. If there are no flights at all taking off from the west coast that day, s/he can't go anyplace at all, much less all the way to the east coast. But what if there are a choice of flights, none of which goes directly to the east coast? Should s/he take the one that goes the farthest towards the east coast or the shortest flight possible? Or is some intermediate distance the best? Or does it make any difference at all?

The Mathematics:

Every group of people working on this problem will come up with different strategies. Here are some of the things that may come up.

1. There is always a fifty-fifty chance that there will be a flight (an arrow—see below) directly from 0 to n. So the probability that you can get from here to there is always at least one-half.



2. If students start by thinking about fairly small values of n, say 4 or 5, they will quickly get bogged down in a swamp of calculations, but they will realize that there are enough similarities between different parts of the computation that there ought to be some way of systematically working through it. They may recognize the utility of working through the smallest problems first.

   For $n = 1$, there is only one arrow involved, the arrow from 0 to 1. It has a fifty-fifty chance of existing, so the probability of being able to reach 1 is .5, and of not being able to reach 1 is 1-.5 = .5, too.

   For $n = 2$, there are three arrows involved: 0 to 2, 0 to 1, and 1 to 2. Each of these arrows exists with a probability of one-half, and there are eight possibilities, each with equal probability of one-eighth. Here is a table for $n = 2$ showing which possibilities will get you from 0 to 2.

Existing arrows	Get from 0 to 2?
none	no
0 -> 2	yes
0 -> 1	no
1 -> 2	no
0 -> 2 and 0 -> 1	yes
0 -> 2 and 1 -> 2	yes
0 -> 1 and 1 -> 2	yes
0 -> 2 and 0 -> 1 and 1 -> 2	yes

   The probability of getting from 0 to 2 is 5/8; of not getting there, 3/8.

3. For $n = 3$, how many arrows are possible in total? There are six: 0 to 3, 0 to 2, 0 to 1, 1 to 3, 1 to 2, and 2 to 3. You could tell ahead of time that there had to be six because each arrow connects a subset of two numbers in the set {0, 1, 2, 3}, and the value of the binomial coefficient ${}_4C_2$ is 6. That means there are $2^6 = 64$ lines in the table enumerating all possibilities. To get to the nice patterns in this problem, there is really no way to avoid this tabulation. There will be 47 possibilities that get you from 0 to 3, and 17 that do not. By the end of this, the students should be trying to invent shortcuts.

   The results for $n = 2, 3$ together might lead to the *conjecture* that the number of possibilities that prevent you from going from 0 to n is a little bit more than 1/4, but there could be various opinions on exactly how little "a little bit" should be.

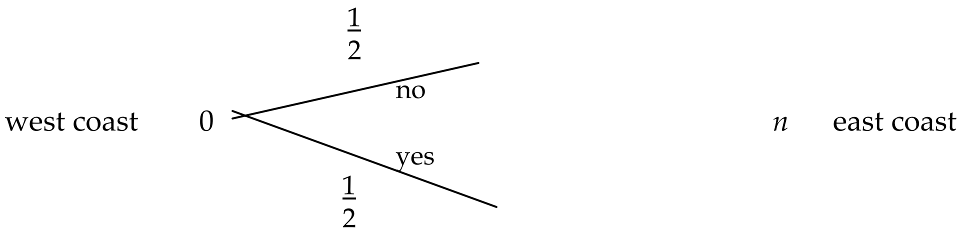
4. You could test this conjecture by doing the computation for $n = 4$. (Now everyone will be motivated to look for shortcuts.) A correct calculation for $n = 4$ will yield the result that there are 1,024 possibilities, 841 of which get you from 0 to 4, and 183 of which do not. Fewer than 1/4 of the possibilities do not get you there, so this demolishes the conjecture from the previous paragraph. But if you observe the following probabilities you might be led to other conjectures.

n	probability of getting from 0 to n
1	1/2
2	5/8 (>1/2)
3	47/64 (>5/8)
4	841/1024 (>3/4)
5	28999/32768 (>7/8)
6	$1928225/2^{21}$

It looks like something is going on here, especially if students use a calculator to change the fractions to decimals, but it is not clear exactly what is happening. By the way, it is true that as n gets larger and larger, the probability of being able to get from 0 to n approaches 1.

5. One thing your students will wrestle with is whether it is more efficient to directly calculate the probability that you can get from 0 to n or the probability that you cannot.

6. In their previous experiences with probability, students may have used tree diagrams as tools. Here is how a useful tree diagram (see following diagrams) can be set up for $n = 4$, beginning with the two branches leading out from the starting point 0. Imagine that you, the traveler, are at 0, the airport on the west coast. You ask, "Could I get to airport 1 if I wanted to go there?" You could if there were an arrow from 0 to 1. What would prevent you from getting to 1 would be the absence of an arrow from 0 to 1 that day, and the probability of the absence is 1/2. So you draw two branches from 0, label the top one "1/2 no." (With probability ½, I can't get from 0 to1.) Label the lower branch "1/2 yes." (With probability ½, I can get from 0 to 1.)

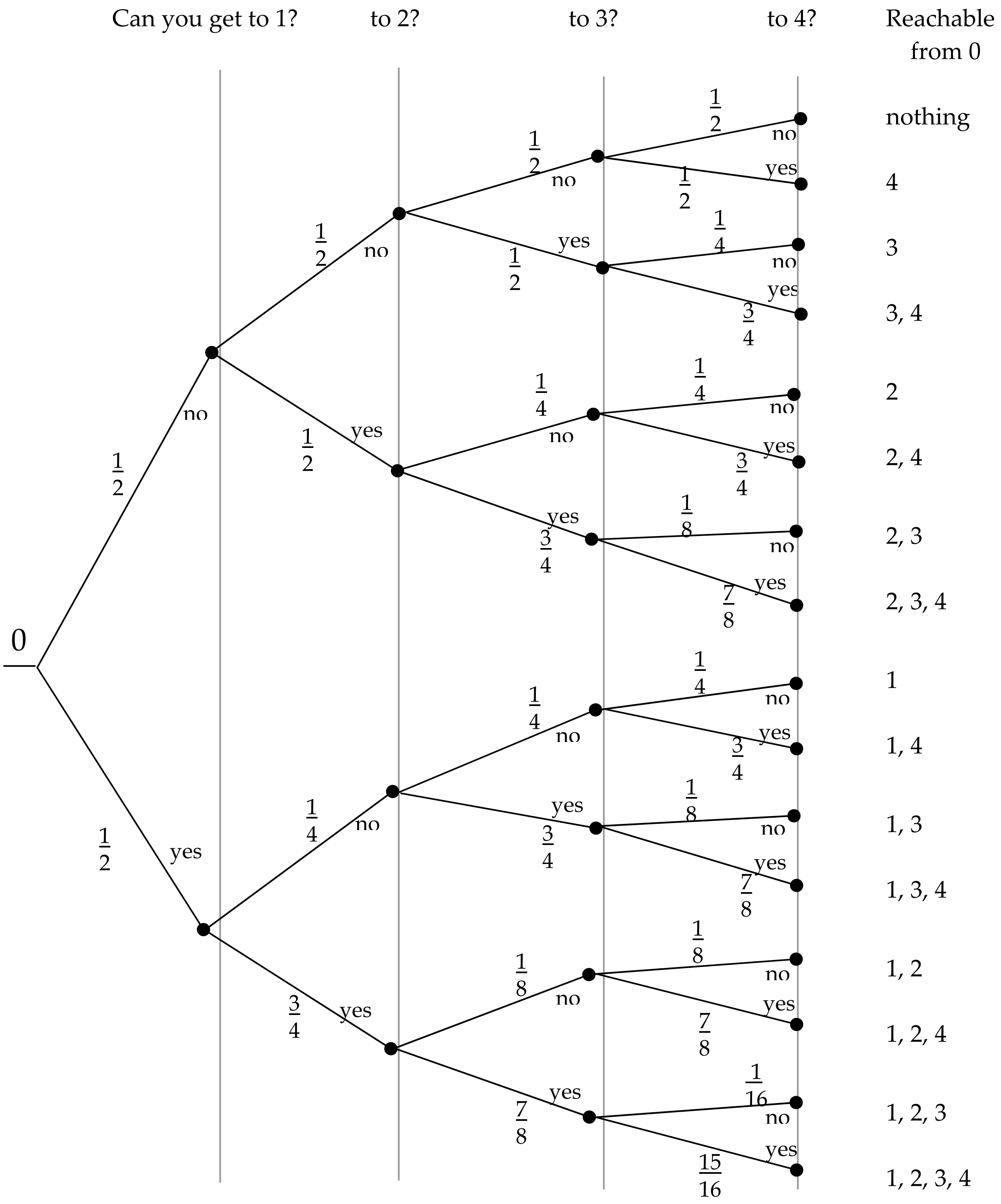


Now you are ready to draw four more branches on your tree—two new ones from the end of each branch that you already have. These branches involve the probability that you can get from 0 to airport 2 today. Let's first see how to get to two branches from the end of the top first branch. If you are at the end of the top first branch, you know that there is no arrow from 0 to 1 today. What will prevent you from reaching airport 2? Only the absence of an arrow from 0 to 2.. The probability of that absence is 1/2. Whether the arrow from 1 to 2 is present today is immaterial, because you can't get to 1 in the first place. So, knowing that there is no arrow from 0 to 1, the probability that you cannot get from 0 to 2 is 1/2 and the probability that you can is also 1/2. So you label the two branches "1/2 no" and "1/2 yes."

What about the two branches that come off the lower first branch? If you are at the end of the lower first branch, then there is an arrow today from 0 to 1. What could prevent you from reaching 2? There would have to be no arrow from 0 to 2 and no arrow from 1 to 2. This is because if there were an arrow from 0 to 2 you would take it. And, if there were an arrow from 1 to 2, you would first go from 0 to 1 and then from 1 to 2. So, you would not be able to get from 0 to 2 only if two arrows were absent, and the probability of that is 1/4, which is (1/2) × (1/2). The probability that you can get from 0 to 2 is 1 - 1/4 = 3/4. So you label these two branches "1/4 no" and "3/4 yes".

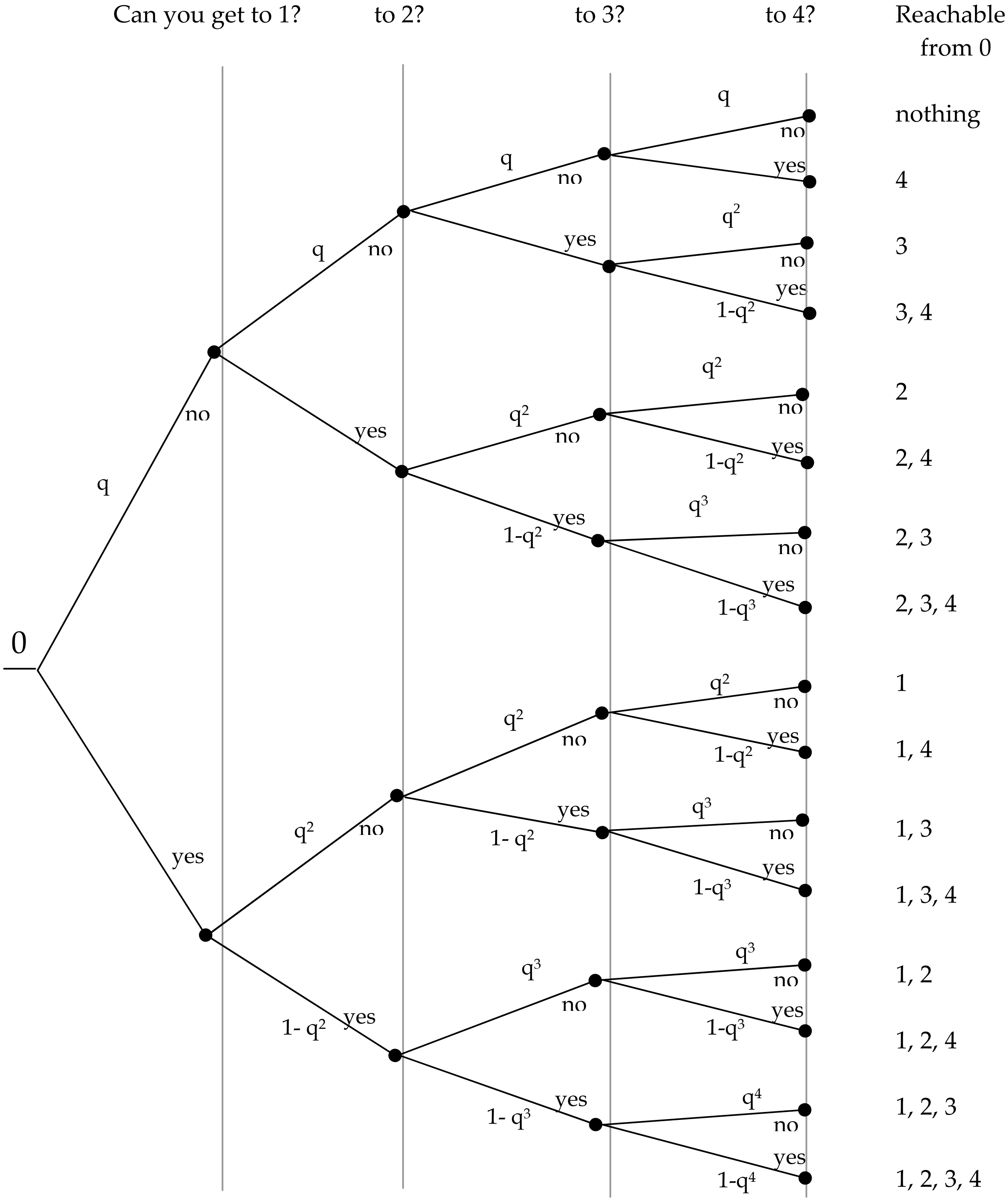
Continue in this way. When the tree is completed, you can trace any path of branches from 0 all the way to the right and the successive yes's and no's that you get tell you which of the airports can be reached from 0. For example, suppose you trace the path that goes up, down, down, up. The labels are 1/2 no, 1/2 yes, 3/4 yes, 1/8 no. This means that you can reach airports 2 and 3 but not 1 or 4, and that the probability of being able to reach just airports 2 and 3 is (1/2) × (1/2) × (3/4) × (1/8).

The probability of being able to reach the East coast when $n = 4$ is just the sum of the probabilities of each path which ends up at a list of airports that includes airport 4. If you add these up carefully for $n = 4$ you will get the probability 841/1024, which appears above in paragraph 4.



Do you see any patterns? Perhaps students will see similarities between subsets with the same number of elements, e.g., the six subsets 3&4, 2&4, 2&3, 1&4, 1&3, and 1&2. What are their probabilities? (Note: Making this tree diagram is probably easier than listing all 1024 possibilities for subsets of the 10 possible arrows.)

To see the similarities between probabilities for subsets with the same number of elements, replace 1/2 by the variable p, let q = 1 – p, and rewrite the tree using q.



Do you see any patterns? Write out the probabilities for subsets of two numbers. Write out the probabilities for subsets of one number. Do you see that being able to sum a finite geometric series is useful?

7. A careful analysis for $n = 5$ will show that the probability of not being able to get from 0 to 5 is

$$q^5 \times [1 + (1 - q^4) + (1 - q^4)(1 - q^3) + (1 - q^4)(1 - q^3)(1 - q^2) + (1 - q^4)(1 - q^3)(1 - q^2)(1 - q)].$$

Do you see how to generalize this to other values of n? Can you check whether the generalization is true for $n = 4, 3, 2, 1$?

Relationships to the Ohio Academic Content Standards, 2002:

Grades 5-7:

Patterns, Functions and Algebra Standard
The student will be able to…

- Describe, extend and determine the rule for patterns and relationships occurring in numeric patterns, computation, geometry, graphs and other applications.
- Represent, analyze and generalize a variety of patterns and functions with tables, graphs, words and symbolic rules.
- Use symbolic algebra to represent and explain mathematical relationships.

Data Analysis and Probability Standard
The student will be able to…

- Interpret data by looking for patterns and relationships, draw and justify conclusions, and answer related questions.
- Find all possible outcomes of simple experiments or problem situations, using methods such as lists, arrays and tree diagrams.
- Collect, organize, display and interpret data for a specific purpose or need.
- Describe the probability of an event using ratios, including fractional notation.
- Evaluate conjectures and predictions based upon data presented in tables and graphs, and identify misuses of statistical data and displays.

Mathematical Processes Standard
The student will be able to…

- Apply and adapt problem-solving strategies to solve a variety of problems, including unfamiliar and non-routine problem situations.
- Use representations to organize and communicate mathematical thinking and problem solutions.

Grades 8-10:

Patterns, Functions and Algebra Standard
The student will be able to…

- Generalize and explain patterns and sequences in order to find the next term and the nth term.
- Use algebraic representations, such as tables, graphs, expressions, functions and inequalities, to model and solve problem situations.

Data Analysis and Probability Standard

The student will be able to…

- Create, interpret and use graphical displays and statistical measures to describe data; e.g., box-and-whisker plots, histograms, scatterplots, measures of center and variability.
- Construct convincing arguments based on analysis of data and interpretation of graphs.
- Compute probabilities of compound events, independent events, and simple dependent events.

Mathematical Processes Standard

The student will be able to…

- Formulate a problem or mathematical model in response to a specific need or situation, determine information required to solve the problem, choose method for obtaining this information, and set limits for acceptable solution.
- Use a variety of mathematical representations flexibly and appropriately to organize, record and communicate mathematical ideas.

Grades 11-12:

Data Analysis and Probability Standard

The student will be able to…

- Create and analyze tabular and graphical displays of data using appropriate tools, including spreadsheets and graphing calculators.

Mathematical Processes Standard

The student will be able to…

- Construct logical verifications or counter-examples to test conjectures and to justify or refute algorithms and solutions to problems.
- Use formal mathematical language and notation to represent ideas, to demonstrate relationships within and among representation systems, and to formulate generalizations.

Relationships to the NCTM Principles and Standards, 2000:

Grades 6-8 and 9-12:

Algebra Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Understand patterns, relations, and functions.
- Represent and analyze mathematical situations and structures using algebraic symbols.
- Use mathematical models to represent and understand quantitative relationships.

Data Analysis and Probability Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them.
- Develop and evaluate inferences and predictions that are based on data.
- Understand and apply basic concepts of probability.

Problem Solving Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Solve problems that arise in mathematics and in other contexts.
- Apply and adapt a variety of appropriate strategies to solve problems.

Representation Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Create and use representations to organize, record, and communicate mathematical ideas.
- Use representations to model and interpret physical, social, and mathematical phenomena.