Ohio's Systemic Initiative Discovery

Coin Flips

One-Page Overview

By Robert B. Brown, The Ohio State University

Topics:

Reasoning, Patterns, Probability, Statistics

Levels:

Grades 5 – 8

Problem:

In this inquiry activity students are asked to investigate whether a sequence of heads and tails was obtained by flipping a coin or by just making a sequence up. This can be a starting point for exploring basic ideas of probability. Parts would be accessible even to very young children, for example, tallying the number of heads that occur in a sequence of coin flips.

Getting Started:

Prepare worksheets on which students can record two sequences of heads and tails, one labeled "pretend flips" and the other labeled "real flips." Give a sheet to each student and have them make up a sequence of heads and tails that they think might realistically come from flipping an honest coin. Have students fill in the "pretend flips" section in order with H's and T's as they make them up.

Ohio Academic Content Standards, 2002

5-7		8-10		11-12	
1. Number, Number Sense and Operations		1. Number, Number Sense and Operations		1. Number, Number Sense and Operations	
2. Measurement		2. Measurement		2. Measurement	
3. Geometry and Spatial Sense		3. Geometry and Spatial Sense		3. Geometry and Spatial Sense	
4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	x	4. Patterns, Functions and Algebra	
5. Data Analysis and Probability	X	5. Data Analysis and Probability	X	5. Data Analysis and Probability	
Mathematical Processes Reasoning		**Mathematical Processes** Reasoning		**Mathematical Processes**	

NCTM Principles and Standards, 2000

6-8		9-12	
1. Number and Operations		1. Number and Operations	
2. Algebra	X	2. Algebra	
3. Geometry		3. Geometry	
4. Measurement		4. Measurement	
5. Data Analysis and Probability	X	5. Data Analysis and Probability	
6. Problem Solving		6. Problem Solving	
7. Reasoning and Proof	X	7. Reasoning and Proof	
8. Communication		8. Communication	
9. Connections		9. Connections	
10. Representation		10. Representation	

Note: Capital X denotes major emphasis; lower case x denotes minor emphasis.

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Topics: Reasoning, Patterns, Probability, Statistics	
Levels: Grades 5 - 8	*Timing:* This activity should be spaced over several days, not necessarily consecutive
Materials: One-page worksheet (7th page of this module) One coin for each student Blackboard	*Prerequisites:* Basic probability Comparing statistical data

Problem:

In this inquiry activity students are asked to investigate whether a sequence of heads and tails was obtained by flipping a coin or by just making up a sequence. This can be a starting point for exploring basic ideas of probability.

Goals:

- Introduce students to the major concepts in probability
- Show how probability can easily solve a seemingly difficult problem
- Provide a sound foundation for the idea of randomness
- Provide a data-gathering situation in which to use statistical tools
- Introduce the idea of a random walk

Big Ideas:

- Randomness
- Sequences
- Probability

Procedure:

1. Prepare worksheets (found at the end of this module) on which students can record two sequences of heads and tails, one labeled "pretend flips" and the other labeled "real flips." Give a sheet to each student and have them make up a sequence of heads and tails they think might realistically come from flipping a coin. Have students fill in the "pretend flips" section in order with H's and T's as they make them up.
2. Next, have each student flip repeatedly a real coin, recording in the "real flips" section what happens.
3. Collect the papers and write one pair of sequences on the board. Choose to write the pretend flips or the real flips first, but don't tell the students which is which. Eventually tell them that the object of the activity is to determine which sequence is real.
4. In groups have them discuss possible ways to tell the sequences apart. After 15 minutes try out their suggestions on the displayed sequences.
5. To tell the sequences apart have the students count runs of heads or tails. How many runs would you expect? Which runs and of what length should you look at? Realize that if you count runs of length 3, then HTTTTH has two runs of TTT, one beginning with the first T in the run of four and the other beginning with the second T. For example, there are eight possible runs of length 3, all of which are equally likely. So in a real sequence of length 65, on average about one-eighth of the 63 runs of length 3 will be TTT, about another eighth will be HTT, and the same for THT, TTH, HHT, and so on. Similar reasoning works for other lengths of runs and sequences.
6. Now comes the mind-numbing part. Boot up your favorite word-processor and transcribe everyone's pair of sequences, calling them sequence I and sequence II, with you randomly choosing whether sequence I is the real or the pretend sequence. Keep a key for yourself, so you will know which is which, but make no indication on the transcription. Put students' initials with their sequences, so they will know whose is whose. You'll be able to get a half dozen or so pairs on one page. Proofread, print, and make copies for all.
7. On the next day bring in the copies, assign to each group of students a few pairs of sequences to work on, and see how good they are at telling the sequences apart using the techniques they discussed earlier or making up new techniques.
8. In a class of 20 persons from middle-school ages to adults, the class will easily identify 70% to 85% of the sequences correctly. Made-up sequences generally have fewer runs of 4 or longer than real sequences. About two-thirds of the made-up sequences will begin with H. In the group of made-up sequences the proportion of H's in each sequence will be more nearly one-half than in the group of real sequences.

Extensions:

After your students have done this experiment, they are no longer naive. Have them do the experiment a second time and see if their made-up sequences are more difficult to detect.

A sequence of heads and tails can take you on a "penny walk." Start at 0 on the number line. If the first flip is a head, go one step to the right to +1. If the first flip is a tail, step left to -1. Continue this way from wherever you are now—each head steps you to the right one unit and each tail steps you left. Such a path produced in response to a random event like flipping a coin is called a random walk. An interesting project would be to determine how many times you should expect to return to 0 in the course of a random walk. (Naturally, the longer the walk, the more times you expect to return to 0. The solution to this problem involves Catalan numbers, whole numbers that turn up in other contexts in mathematics.

The Mathematics:

The students will probably guess that counting runs of heads and tails can lead to making good guesses about which sequence is real. After some discussion they will probably admit that in making up sequences they tried not to make the runs very long, because they are unlikely. However, a sequence of length 65 is long enough that the expected number of runs of length 6 is about two.

In considering runs, there are a few important ideas: What are all the possible patterns of heads and tails in a section of a given length in sequence, such as length 2, length 3, etc.? Here are all of the eight possibilities for length three:

HHH HHT HTH THH HTT THT TTH TTT

Are all these patterns equally likely? Yes, because a flipped coin has no memory; each of the sequences shown is just as likely as any of the others.

How do you count how many times a given pattern occurs in a long sequence? This should be discussed in order to resolve even such a simple question of how many runs of three heads are in a run of four heads. Here is an approach. Focus on any sequence of three flips within a long sequence of flips, say the flips in positions 15, 16, and 17. Now imagine a whole book full of long sequences generated by flipping a real coin. Look at the same positions 15, 16, and 17 in each sequence. Each of the eight possibilities has an equally likely chance of filling the three positions. Therefore, the chance that any particular sequence would fill those positions is one-eighth. But this argument has an unexpected result. Pick any particular possibility, such as HTH, and look at all possible starting positions for three consecutive flips. For example, in a sequence of total length 65, three consecutive flips could begin in positions 1, 2, 3, 4, and so on, up to position 63, which would be the starting point of the last three consecutive flips. This gives 63 starting positions and the chances are one-eighth that the possibility HTH begins in any given starting position. So the number of HTH's that you would expect to find in a sequence is about 63 x (1/8), which is about 8. In a sequence of total length 65 you would expect to count about 8 HHH's too.

For possibilities of length 4, such as HHHH, you would expect to count about 4. This is because there are 16 possibilities of length 4, and in 62 possible starting places you would expect 62 x (1/16), which is about 4. It is important to see that in a sequence such as THHHHT you count one run of four H'S and two runs of three H'S, because a run of three H'S starts in position 2 and another in position 3.

About how many times should you expect to count a given pattern in a long sequence? If the given pattern has length k, it is just one of the many possibilities for patterns of length k. The number of such patterns is 2^k, and all of them are equally likely (if generated by a randomly-flipped honest coin). In a sequence of total length n, there are $n - k + 1$ starting positions for a sequence of length k, so that the number of occurrences of the given pattern that you would expect would be $(n - k + 1)/ 2^k$.

Here is a sample pair of sequences of 65 flips. See if you can discover which one is made up before reading on.

<u>Sequence I</u>

HTHTHHTHHH HHHHTHHTH THTHTTHHHT TTHHHHTHHH
THTHTTHHHT HTHTHTHTTH THHTT

<u>Sequence II</u>

HHTHTTTHTH HTTHTHTTHH TTHHHTHTTH HTTTTHTHHT
THTHHTTTHT HTHHTHTTHT HTHTT

Pattern	# Expected if random	Sequence I	Sequence II
T	65/2 (about 32)	25	35
H	65/2 (about 32)	38	30
HH	64/4 (about 16)	19	9
HT	64/4 (about 16)	20	21
TH	64/4 (about 16)	19	19
TT	64/4 (about 16)	6	14
HHH	63/8 (about 8)	10	1
HTH	63/8 (about 8)	15	10
TTT	63/8 (about 8)	1	4
HHHH	62/16 (about 4)	5	0
TTTT	62/16 (about 4)	0	1
HHHHH	61/32 (about 2)	3	0
HHTHH	61/32 (about 2)	3	0
THHTT	61/32 (about 2)	1	5
Changes from H to T or vice versa	64/2 (about 32)	38	41

Sequence I is the real one, and sequence II is made up.

Some Statistics:

After the real sequence from the pair has been identified for the class, have each student count the number of heads in his or her real and made-up sequences. Make a histogram of the numbers of heads in all the real sequences. Make another histogram for the made-up sequences. You will notice a startlingly sharp peak in the histogram for made-up sequences. Use a box-and-whiskers plot and calculate the mean and standard deviation for the real and for the made-up sequences. Count what proportion of the made-up sequences start with a head.

Ohio's Systemic Initiative Discovery

Name: ________________________

Pretend Flips:

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
11.	12.	13.	14.	15.	16.	17.	18.	19.	20.
21.	22.	23.	24.	25.	26.	27.	28.	29.	30.
31.	32.	33.	34.	35.	36.	37.	38.	39.	40.
41.	42.	43.	44.	45.	46.	47.	48.	49.	50.
51.	52.	53.	54.	55.	56.	57.	58.	59.	60.
61.	62.	63.	64.	65.					

Real Flips:

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
11.	12.	13.	14.	15.	16.	17.	18.	19.	20.
21.	22.	23.	24.	25.	26.	27.	28.	29.	30.
31.	32.	33.	34.	35.	36.	37.	38.	39.	40.
41.	42.	43.	44.	45.	46.	47.	48.	49.	50.
51.	52.	53.	54.	55.	56.	57.	58.	59.	60.
61.	62.	63.	64.	65.					

Relationships to the Ohio Academic Content Standards, 2002:

Grades 5-7:

Patterns, Functions and Algebra Standard
The student will be able to…

- Describe, extend and determine the rule for patterns and relationships occurring in numeric patterns, computation, geometry graphs and other applications.

Data Analysis and Probability Standard
The student will be able to…

- Interpret data by looking for patterns and relationships, draw and justify conclusions, and answer related questions.
- Find all possible outcomes of simple experiments or problem situations, using methods such as lists, arrays and tree diagrams.
- Describe the probability of an event using ratios, including fractional notation.
- Make and justify predictions based on experimental and theoretical probabilities.

Mathematical Processes Standard
The student will be able to…

- Use deductive thinking to construct informal arguments to support reasoning and to justify solutions to problems.
- Use inductive thinking to generalize a pattern of observations for particular cases, make conjectures, and provide supporting arguments for conjectures.

Grades 8-10:

Data Analysis and Probability Standard
The student will be able to…

- Compute probabilities of compound events, independent events, and simple dependent events.
- Make predictions based on theoretical probabilities and experimental results.

Mathematical Processes Standard
The student will be able to…

- Apply reasoning processes and skills to construct logical verifications or counter-examples to test conjectures and to justify and defend algorithms and solutions.

Relationships to the NCTM Principles and Standards, 2000:

Grades 6-8:

Algebra Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Understand patterns, relations, and functions.

Data Analysis and Probability Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Develop and evaluate inferences and predictions that are based on data.
- Understand and apply basic concepts of probability.

Reasoning and Proof Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Make and investigate mathematical conjectures.