

How to Fix an Unfair Game

One-Page Overview

By Robert B. Brown, The Ohio State University

Topics:

Probability, Expected Values

Levels:

Grades 5 – 8

Problem:

Michelle and Jim wager on picking colored chips out of a bag. In the bag are two black chips and one red one. Someone takes out a chip, notes its color, and puts it back. Then he or she reaches in again and takes out a chip. If it is the same color as the first one, Michelle wins a dollar from Jim. If not, Jim wins a dollar from Michelle. Is the game fair to both Michelle and Jim?

Getting Started:

Pair off the students and give each pair a bag containing one red chip and two black chips. Ask each player to start with an imagined money supply of ten dollars and to keep track of their money on a sheet of paper as they play.

Ohio Academic Content Standards, 2002

5-7		8-10		11-12	
1. Number, Number Sense and Operations	x	1. Number, Number Sense and Operations	x	1. Number, Number Sense and Operations	
2. Measurement		2. Measurement		2. Measurement	
3. Geometry and Spatial Sense		3. Geometry and Spatial Sense		3. Geometry and Spatial Sense	
4. Patterns, Functions and Algebra		4. Patterns, Functions and Algebra		4. Patterns, Functions and Algebra	
5. Data Analysis and Probability	X	5. Data Analysis and Probability	X	5. Data Analysis and Probability	
Mathematical Processes Problem Solving Representation Communication		Mathematical Processes Problem Solving Representation Communication		Mathematical Processes	

NCTM Principles and Standards, 2000

6-8		9-12	
1. Number and Operations	x	1. Number and Operations	
2. Algebra		2. Algebra	
3. Geometry		3. Geometry	
4. Measurement		4. Measurement	
5. Data Analysis and Probability	X	5. Data Analysis and Probability	
6. Problem Solving	X	6. Problem Solving	
7. Reasoning and Proof		7. Reasoning and Proof	
8. Communication	x	8. Communication	
9. Connections		9. Connections	
10. Representation	X	10. Representation	

Note: Capital X denotes major emphasis; lower case x denotes minor emphasis.

How to Fix an Unfair Game

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<u>Topics:</u> Probability, Expected values	
<u>Levels:</u> Grades 5 – 8	<u>Timing:</u> Two hours
<u>Materials:</u> A bag with one red and two black chips for each pair of students	<u>Prerequisites:</u> Operations with fractions

Problem:

Michelle and Jim wager on picking colored chips out of a bag. In the bag are two black chips and one red one. Someone takes out a chip, notes its color, and puts it back. Then he or she reaches in again and takes out a chip. If it is the same color as the first one, Michelle wins a dollar from Jim. If not, Jim wins a dollar from Michelle. Is the game fair to both Michelle and Jim?

Goals:

- Discover a notion of a “fair game”
- Develop expected value as an average
- See how adjusting the payoffs of a game affect expected values
- Practice using tree diagrams

Big Ideas:

- Fair game
- Expected value
- Probability
- Tree diagrams

Procedure:

1. Two players wager on picking colored chips out of a bag. In the bag are two black chips and one red one. Someone takes out a chip, notes its color, and puts it back. Then he or she reaches in again and takes out a chip. If it is the same color as the first one, Michelle wins a dollar from Jim. If not, Jim wins a dollar from Michelle. Pair off the students and give each pair a bag containing one red and two black chips. Ask each player to start with an imagined money supply of ten dollars and to keep track of their money on a sheet of paper as they play. Let them play for ten minutes, and then report whether they think the game is fair or not.
2. Join pairs together to make groups of four students. Let them discuss in their groups what they mean by a fair game. Then let each group report, and see if the class can agree on the concept of a fair game. (The usual notion is that a game is fair if neither side has a long-term advantage on average.)
3. After the concept of a fair game is decided upon, let the small groups analyze the probability that Michelle wins and the probability that Jim wins.
4. In small groups, let them discuss how the game might be modified to make it a fair game. Have the group report. You can always have them play some of the modified games and report their findings.

Extensions:

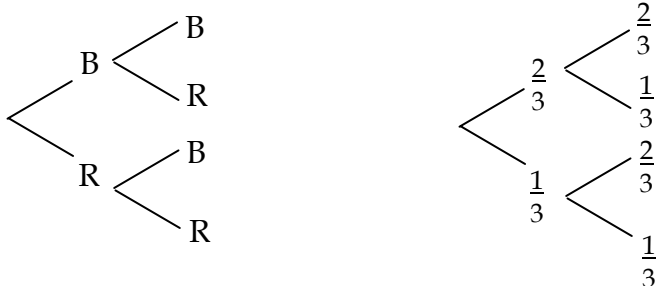
1. Here are some other games that could be used to illustrate the same ideas.
 - (a) Three colored chips. This involves three chips with a different color on each side. For example, with colors red, white, and blue, one chip is red-white, another is white-blue, and the third is blue-red. The chips are tossed up and fall to the table. Player A wins a dollar if any sides that show match in color. Player B wins a dollar if the sides show all different colors.
 - (b) Chuck-a-Luck. This game involves rolling three dice once. You pay a dollar to play a game. After the dice are rolled you get back a dollar for each die that shows a 6.
2. Let the students propose their own games.

The Mathematics:

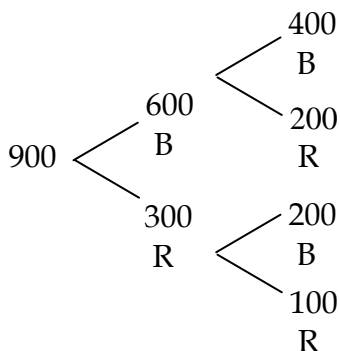
The theoretical analysis of the probabilities of the game Michelle and Jim played can be made using a tree diagram. After the probability analysis is done, the wagers have to be brought in.

For each of the two selections, there are two black chips and one red to choose from. Each chip is equally likely to be chosen. Therefore, at each selection, the probability is $\frac{2}{3}$ that the choice will be black, $\frac{1}{3}$ that it will be red.

Below are two tree diagrams. The one on the left shows the possible outcomes of the two selections. Thus, the top branch of the tree followed to its right-hand end represents black for the first selection followed by black for the second. The right-hand tree is the same diagram but with each piece labeled with the probability of going along that piece.

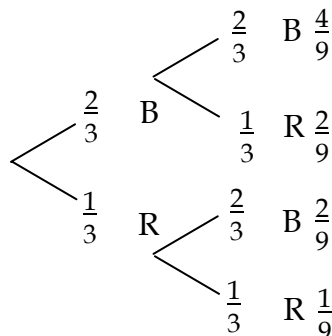


The third tree, shown below, shows what happens in theory if you play the game 900 times. At the first stage you pick black 600 times and red 300 times. If you first picked black, then at the second stage you will pick black again 400 times and red 200 times. If you first picked red, then at the second stage you will pick black 200 times and red 100 times.



Altogether, the number of times that you will get the same color on both selections is 500; 400 from black-black and 100 from red-red. The colors match 500 out of 900 times so the probability of Michelle winning is $\frac{5}{9}$. The probability of Jim winning is only $\frac{4}{9}$. Therefore, the game is not fair, if by fair you mean that on average the two players will both break even.

Below is a fourth tree diagram which is the culmination of the use of tree diagrams in this problem. It is the same as the first right-hand diagram except that the final end of each branch is labeled with the probability of ending up there. You see that the probability at each of the four ends is the product of the probabilities along all of the pieces it takes to get there. Furthermore, Michelle's $\frac{5}{9}$ probability of winning is the sum of the top $\frac{4}{9}$ and the bottom $\frac{1}{9}$, while Jim's $\frac{4}{9}$'s is the sum of the two $\frac{2}{9}$'s in the middle.



Now the second step, considering the wagers. To do this we calculate the average amount won per game by one of the players, say, Michelle. In technical terms this is the expected value of Michelle's winnings. To do this let's go back to what happens when we play the game 900 times. Michelle wins 500 times and wins a total of \$500. But Jim wins 400 times and wins \$400. Thus, over the 900 plays, Michelle's net winnings are $\$500 - \$400 = \$100$. The expected value of Michelle's winning is $\$100/900 = 11 \frac{1}{9}$ cents. The game is not fair. This expected amount can be obtained directly from the probabilities without imagining 900 games in the following way. Observe that the average winnings for Michelle were calculated as follows:

$$(\$500 - \$400)/900.$$

This is the same as

$$(\frac{500}{900}) \times \$1.00 - (\frac{400}{900}) \times \$1.00$$

which is equal to: $(\frac{5}{9}) \times (\$1.00) + (\frac{4}{9}) \times (-\$1.00)$.

It is this last line that involves the probabilities together with the payoffs to Michelle. For each outcome (there are two—Michelle wins and Michelle loses), multiply together the probability of that outcome and the payoff to Michelle if that outcome occurs. Add the products and you have the expected value of Michelle's winning (per game).

The easy way to modify the game to make it fair is to let Jim win more than a dollar every time he wins, but leave Michelle's win at a dollar. All you need to do is increase Jim's winning to balance the \$500 that Michelle wins. Jim wins 400 times and if he wins \$1.25 each time, he will get \$500, just like Michelle. So this simple modification calls for Michelle to get a dollar from Jim each time she wins, but for Jim to get \$1.25 from Michelle each time that he wins.

Statistics:

Even if the game is modified and a fair version is played, the two players cannot expect to be even all of the time. Because of randomness it is quite possible for one player to pull far ahead. Eventually the one behind will pull even again, but that could take a long time. Before that happens, the players could run out of time or the loser could exhaust his or her stake.

Acknowledgement

This game came from Meyer, Channel, & Browning (1992), *Probability for Elementary Teachers*.

Relationships to the Ohio Academic Content Standards, 2002:

Grades 5-7:

Number, Number Sense and Operations Standard

The student will be able to...

- Use order of operations, including use of parenthesis and exponents to solve multi-step problems, and verify and interpret the results.

Data Analysis and Probability Standard

The student will be able to...

- Find all possible outcomes of simple experiments or problem situations, using methods such as lists, arrays and tree diagrams.
- Describe the probability of an event using ratios, including fractional notation.
- Make and justify predictions based on experimental and theoretical probabilities.

Mathematical Processes Standard

The student will be able to...

- Apply and adapt problem-solving strategies to solve a variety of problems, including unfamiliar and non-routine problem situations.
- Use representations to organize and communicate mathematical thinking and problem solutions.
- Communicate mathematical thinking to others and analyze the mathematical thinking and strategies of others.

Grades 8-10:

Data Analysis and Probability Standard

The student will be able to...

- Compute probabilities of compound events, independent events, and simple dependent events.
- Make predictions based on theoretical probabilities and experimental results.

Mathematical Processes Standard

The student will be able to...

- Formulate a problem or mathematical model in response to a specific need or situation, determine information required to solve the problem, choose a method for obtaining this information, and set limits for acceptable solution.
- Use a variety of mathematical representations flexibly and appropriately to organize, record and communicate mathematical ideas.

Relationships to the NCTM Principles and Standards, 2000:

Grades 6-8:

Number and Operations Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to...

- Understand meanings of operations and how they relate to one another.

Data Analysis and Probability Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to...

- Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them.
- Understand and apply basic concepts of probability.

Problem Solving Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to...

- Build new mathematical knowledge through problem solving.
- Apply and adapt a variety of appropriate strategies to solve problems.

Communication Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to...

- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.

Representation Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to...

- Create and use representations to organize, record, and communicate mathematical ideas.
- Select, apply, and translate among mathematical representations to solve problems.
- Use representations to model and interpret physical, social, and mathematical phenomena.