Ohio's Systemic Initiative
Discovery

Pascal’s Triangle I

One-Page Overview

By Robert B. Brown, The Ohio State University

Topics:

Pascal’s Triangle, Patterns

Levels:

Grades 6 – 8

Problem:

Construct Pascal’s triangle. Find patterns connected with it.

Getting Started:

Exhibit staggered rows of zeros—an ocean of zeros:

```
0  0  0  0  0  0  0  0  0  0  0  0
  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0
```

Notice that each number in the ocean is the sum of the two neighbors just above it. Now drop a single 1 into the ocean and see how this summing generates Pascal’s triangle.

```
0  0  0  0  0  0  0  0  0  0  0  0
  0  0  0  0  0  0  1  0  0  0  0  0
0  0  0  0  0  0  1  1  0  0  0  0
  0  0  0  0  0  1  2  1  0  0  0  0
0  0  0  0  0  1  3  3  1  0  0  0
```

Ohio Academic Content Standards, 2002

5-7		8-10		11-12	
1. Number, Number Sense and Operations	X	1. Number, Number Sense and Operations	x	1. Number, Number Sense and Operations	
2. Measurement		2. Measurement		2. Measurement	
3. Geometry and Spatial Sense		3. Geometry and Spatial Sense		3. Geometry and Spatial Sense	
4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	
5. Data Analysis and Probability	x	5. Data Analysis and Probability	X	5. Data Analysis and Probability	
Mathematical Processes Problem Solving Representation		**Mathematical Processes** Problem Solving Representation		**Mathematical Processes**	

NCTM Principles and Standards, 2000

6-8		9-12	
1. Number and Operations	X	1. Number and Operations	
2. Algebra	X	2. Algebra	
3. Geometry		3. Geometry	
4. Measurement		4. Measurement	
5. Data Analysis and Probability	x	5. Data Analysis and Probability	
6. Problem Solving	x	6. Problem Solving	
7. Reasoning and Proof		7. Reasoning and Proof	
8. Communication		8. Communication	
9. Connections		9. Connections	
10. Representation	X	10. Representation	

<u>Note</u>: Capital X denotes major emphasis; lower case x denotes minor emphasis.

Pascal's Triangle I

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Topics: Pascal's triangle, Patterns	
Levels: Grades 6 – 8	*Timing:* 1–3 class periods. The first period should be sufficient to simply construct Pascal's triangle. More time can be used to examine patterns and show connections between Pascal's triangle and other ideas in mathematics.
Materials: None	Prerequisites: None

Problem:

Construct Pascal's triangle. Find patterns connected with it.

Goals:

- Construct Pascal's triangle
- Discover and explain simple patterns
- Introduce the notation ${}_nC_k$ for counting the number of subsets

Big Ideas:

- Patterns
- The notation ${}_nC_k$

Procedure:

1. Exhibit staggered rows of zeros—an ocean of zeros:

   0 0 0 0 0 0 0 0 0 0 0 0

   0 0 0 0 0 0 0 0 0 0 0 0

   0 0 0 0 0 0 0 0 0 0 0 0

   0 0 0 0 0 0 0 0 0 0 0 0

   0 0 0 0 0 0 0 0 0 0 0 0

   Notice that each number in the ocean is the sum of the two neighbors just above it.

   Now drop a single 1 into the ocean and see how this summing generates Pascal's triangle.

   0 0 0 0 0 0 0 0 0 0 0 0

   0 0 0 0 0 0 1 0 0 0 0 0

   0 0 0 0 0 0 1 1 0 0 0 0

   0 0 0 0 0 1 2 1 0 0 0 0

   0 0 0 0 0 1 3 3 1 0 0 0

2. What is the sum of the numbers of a given row in Pascal's triangle? Find the sum for a few rows. Do you see a pattern? Make a conjecture.

3. What if you add the numbers in a row with alternating signs? For example, in the fourth row (the top row is *row 0*) you would get 1 + 4 – 6 + 4 – 1. Any pattern? Make a conjecture.

4. In Pascal's triangle underline the entries that are divisible by 3. Any pattern?

   Underline the numbers that are divisible by 5. Any pattern?

   To examine patterns involving numbers divisible by larger numbers, you will have to build a huge Pascal's triangle. That could be a class project. Don't forget to have the students double check everything.

5. What if you add the squares of the numbers in a given row?

6. If your students can multiply polynomials, ask them to find $(x + y)^2$, $(x + y)^3$, $(x + y)^4$, etc. Do you see where the coefficients of the expanded polynomial appear in Pascal's triangle?

7. Observe that for small n and k we can count the number of different subsets of size k in a set with n elements, written ${}_nC_k$ (see the Pascal's Triangle II module for development of this idea). For any that you find, you can see that these are also equal to the numbers ${}_nC_k$ which are found as the k^{th} *entry* of the n^{th} *row* of Pascal's triangle (the first row is *row 0* and the first entry of each row is *entry 0*). Of course, this raises the issue of whether these numbers are the same for all n and k. Seeing that they are the same can be a subject of inquiry, too, but for now it might be more fun to just play with Pascal's triangle and look for patterns.

   Do you see a pattern in the numbers ${}_0C_0$, ${}_1C_0$, ${}_2C_0$, ${}_3C_0$,...(the list of 0^{th} entries of each row)? What about ${}_1C_1$, ${}_2C_1$, ${}_3C_1$, ${}_4C_1$, ... (the list of 1^{st} entries of each row)? Do you recognize the ${}_2C_2$, ${}_3C_2$, ${}_4C_2$, ${}_5C_2$, ... (the list of 2^{nd} entries of each row)?

<u>*Extensions:*</u>

There are many more patterns in Pascal's triangle. If you pose to the small groups the question of looking for patterns in Pascal's triangle, the students may well find some interesting ones themselves.

Many other interesting patterns and relationships can be readily found in articles in encyclopedias and journals, in mathematical puzzle books, and on the Internet.

The Mathematics:

Remember that numbers ${}_nC_k$ are found as the k^{th} *entry* of the n^{th} *row* of Pascal's triangle with the first row being *row 0* and the first entry of each row being *entry 0*.

The sum of any row of Pascal's triangle (Procedure #2) is a power of 2, specifically 2^n for the n^{th} *row*. The first row, which is the row (1 1), adds up to 2 itself, the next row (1 2 1) adds up to $4 = 2^2$, and so on. Even the 0^{th} row adds up to $2^0 = 1$. To see the general fact, you just start with the top row (1) and see that the sum of each successive row is double that of the preceding row. To see this, think of how you form the numbers in a row of Pascal's triangle. You get each individual number by adding together the two numbers just above it. So if you add together all of the numbers, you are adding in twice each number from the row above, once for the number below it to the left and once for the number below it to the right. The 1's at the end of the row above only get added in once each, because each of them has only one number below it. But the two new 1's in the row below get added in, too, and that makes up for using each 1 above only once.

If you add up a row of Pascal's triangle with alternating signs you get 0 (Procedure #3). This is elementary for the odd-numbered rows. Each odd-numbered row has an even number of numbers, and because of the symmetry they come in equal pairs with opposite signs and cancel each other. This kind of reasoning will not do for the even numbered rows. In the fourth row (1 4 6 4 1) you get $1 - 4 + 6 - 4 + 1 = 0$, but the reason is somewhat deeper. If you expand the polynomial $(1 + x)^n$ into powers of x, you get the n^{th} *row* of Pascal's triangle as coefficients (as explained below in the section on coefficients of $(x + y)^n$):

$$(1 + x)^n = {}_nC_0 + {}_nC_1x + {}_nC_2x^2 + \ldots + {}_nC_nx^n.$$

You still have a valid equation if you substitute any number for x. If you substitute $x = -1$ you get $0 = {}_nC_0 - {}_nC_1 + {}_nC_2 + \ldots +(-1)^n{}_nC_n$, and so you've shown that alternating the signs gives a sum of 0 whether you have an even or an odd number of terms. Interestingly, if you substitute $x = +1$, you get $(1 + 1)^n = 2^n = {}_nC_0 + {}_nC_1 + {}_nC_2 + \ldots + {}_nC_n$, the result explained earlier for Procedure #2.

If you underline the numbers that are divisible by a fixed prime such as 3 or 5, you see a pattern of triangles within Pascal's triangle (Procedure #4).

You can see why the coefficients of $(x + y)^n$ (Procedure #6) are the numbers in Pascal's triangle by seeing how the coefficients of $(x + y)^n$ are obtained from the coefficients of $(x + y)^{n-1}$ by multiplying the expansion of $(x + y)^{n-1}$ by $(x + y)$. You will see that each coefficient of $(x + y)^n$ is the sum of two consecutive coefficients of $(x + y)^{n-1}$ in exactly the same way that each entry in Pascal's triangle is the sum of the two neighboring entries in the previous row. This means that you can get the coefficients of $(x + y)^n$ just by going through the Pascal's triangle construction, so:

$$(x + y)^n = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \ldots. + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n$$

For the last question in Procedure #7, ${}_2C_2 = 1$, ${}_3C_2 = 3$, ${}_4C_2 = 6$, ${}_5C_2 = 10$, ${}_6C_2 = 15$, ... are the sums of all the consecutive whole numbers starting at 1 and ending at a specific whole number. For example, ${}_3C_2 = 3 = 1 + 2$ and ${}_4C_2 = 6 = 1 + 2 + 3$. Why would this happen?

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1

Relationships to the Ohio Academic Content Standards, 2002:

Grades 5-7:

Number, Number Sense and Operations Standard
The student will be able to…

- Use order of operations, including use of parenthesis and exponents to solve multi-step problems, and verify and interpret the results.
- Use a variety of strategies, including proportional reasoning, to estimate, compute, solve and explain solutions to problems involving integers, fractions, decimals and percents.

Patterns, Functions and Algebra Standard
The student will be able to…

- Describe, extend and determine the rule for patterns and relationships occurring in numeric patterns, computation, geometry, graphs and other applications.
- Use rules and variables to describe patterns, functions and other relationships.

Mathematical Processes Standard
The student will be able to…

- Apply and adapt problem-solving strategies to solve a variety of problems, including unfamiliar and non-routine problem situations.
- Select, apply, and translate among mathematical representations to solve problems; e.g., representing a number as a fraction, decimal or percent as appropriate for a problem.

Grades 8-10:

Number, Number Sense and Operations Standard
The student will be able to…

- Apply properties of operations and the real number system, and justify when they hold for a set of numbers.

Patterns, Functions and Algebra Standard
The student will be able to…

- Generalize and explain patterns and sequences in order to find the next term and the nth term.
- Use algebraic representations, such as tables, graphs, expressions, functions and inequalities, to model and solve problem situations.

Data Analysis and Probability Standard

The student will be able to…

- Use counting techniques, such as permutations and combinations, to determine the total number of options and possible outcomes.

Mathematical Processes Standard

The student will be able to…

- Formulate a problem or mathematical model in response to a specific need or situation, determine information required to solve the problem, choose a method for obtaining this information, and set limits for acceptable solution.
- Use a variety of mathematical representations flexibly and appropriately to organize, record, and communicate mathematical ideas.

Relationships to the NCTM Principles and Standards, 2000:

Grades 6-8:

Number and Operations Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Understand the meanings of operations and how they relate to one another.

Algebra Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Use mathematical models to represent and understand quantitative relationships.

Problem Solving Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Build new mathematical knowledge through problem solving.

Representation Standard

Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Select, apply, and translate among mathematical representations to solve problems.