Ohio's Systemic Initiative
Discovery

Pool Table Geometry

One-Page Overview

By Robert B. Brown, The Ohio State University

Topics:

Problem solving, Number relations

Level:

Grades 7 - 12

Problem:

Have students think of questions to ask about the following situation.

Getting Started:

Draw on squared paper or build with snap cubes a rectangular pool table. Trace out the path of a ball until it hits a corner. You must start the ball at a corner and use only 45-degree angles.

Ohio Academic Content Standards, 2002

5-7		8-10		11-12	
1. Number, Number Sense and Operations	X	1. Number, Number Sense and Operations	X	1. Number, Number Sense and Operations	x
2. Measurement		2. Measurement		2. Measurement	
3. Geometry and Spatial Sense		3. Geometry and Spatial Sense		3. Geometry and Spatial Sense	
4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	x
5. Data Analysis and Probability		5. Data Analysis and Probability		5. Data Analysis and Probability	
Mathematical Processes Problem Solving Connections		**Mathematical Processes** Problem Solving Connections		**Mathematical Processes** Problem Solving Connections	

NCTM Principles and Standards, 2000

6-8		9-12	
1. Number and Operations	X	1. Number and Operations	x
2. Algebra	X	2. Algebra	X
3. Geometry		3. Geometry	
4. Measurement		4. Measurement	
5. Data Analysis and Probability		5. Data Analysis and Probability	
6. Problem Solving	X	6. Problem Solving	X
7. Reasoning and Proof		7. Reasoning and Proof	
8. Communication		8. Communication	
9. Connections	x	9. Connections	x
10. Representation		10. Representation	

Note: Capital X denotes major emphasis; lower case x denotes minor emphasis.

Pool Table Geometry

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Topics: Problem solving, Number relations	
Levels: Grades 7 - 12	*Timing:* 1-2 hours
Materials: Squared paper (1/2 inch or larger)	*Prerequisites:* Least common multiples, common divisors

Problem:

Draw on squared paper or build with snap cubes a rectangular pool table. Trace out the path of a ball until it hits a corner. You must start the ball at a corner and use only 45-degree angles. Have students think of questions to ask about the situation.

Goals:

- See Least Common Multiples and Greatest Common Divisors in a physical setting
- Practice problem solving

Procedure:

1. Individually or in small groups draw on squared paper or build with snap cubes a rectangular pool table. Trace the path of a ball until it hits a corner. You must start the ball at a corner and use only 45-degree angles. Think of some questions that could be asked about this activity.
2. Convene the class as a whole to discuss the questions that could be asked about this activity.
3. Samples of questions that have been posed by students are:
    a. How many ricochets does it take before the ball hits a corner?
    b. Do ricochets form a square pattern?
    c. Is there any relationship between the lengths of the sides of the rectangle and the number of ricochets that it takes to get to a corner?
    d. Does it matter whether the sides of the pool table measure an even or an odd number of units?
    e. How many tiles does the ball touch until it goes into a pocket at a corner?
    f. Can you make any paths that repeat exactly?
    g. Is there a relationship between the number of times that a ball hits a side and the length of that side?
    h. Is the pattern affected by replacing a 3 × 5 table by a 5 × 3 table?
    i. If we know the number of ricochets and the number of tiles touched, can we predict the lengths of the sides of the table?
    j. Can we predict which corner the ball ends at if we know the lengths of the sides?

Extensions

Is it worthwhile to look at angles other than 45 degrees?

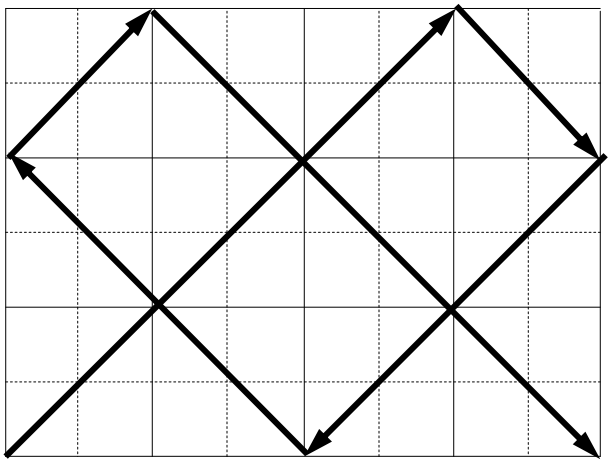
The Mathematics

Look at the fifth question: How many tiles does the ball touch until it goes into a pocket at a corner? Here are some simple observations about that question.

* On a 4 × 6 table a ball will hit 12 tiles.
* If the table is a perfect $n \times n$ square, then the ball hits exactly n tiles.
* On a 4 × 8 table the ball hits 8 tiles.

Observations such as these might lead the students to suppose that the number of tiles hit is the least common multiple (LCM) of the length and width of the rectangular table. Why should it be the LCM?

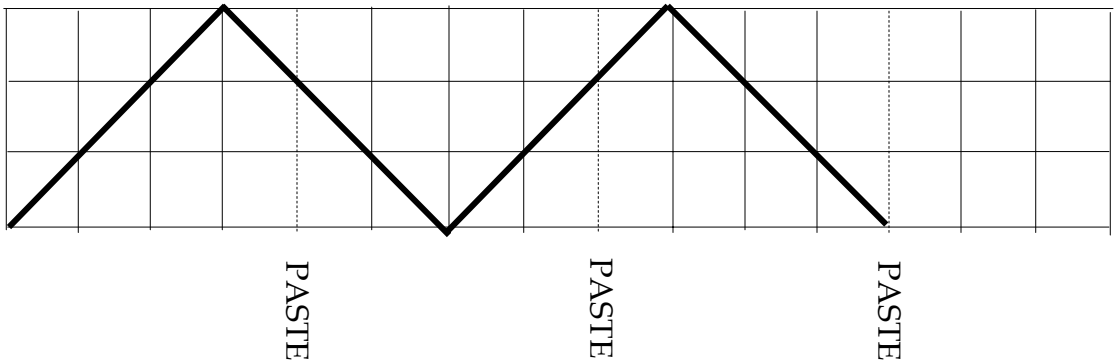
One way to attack the problem is to find the largest number that divides into both the length and the width. That number is the greatest common divisor (GCD) of the length and the width. Then you can divide the entire table into larger squares that are the GCD on a side. The ball will travel diagonally across each such square hitting exactly the GCD number of tiles in each square. Here is a picture for a 6 × 8 table. The GCD is 2 in this case, so the large rectangle is divided into 2 × 2 squares.



Start Finish

You can just as easily investigate the question for the 3 × 4 table of large squares if you remember that the number of tiles hit is twice the number of large squares hit. (Remember to multiply by 2 because the GCF of 6 and 8 is 2.)

Using the large square as the unit of measurement has the advantage that the length and width will have no common factor other than 1. You can predict when the ball first hits a corner by pasting together end-to-end as many copies of the table as you need, and taking out the edges of the table that run vertically. Then the ball, instead of bouncing off of a vertical edge, simply crosses that edge and goes on into the next copy of the table. Here is a paste-up for the 3 × 4 table of large squares.



You find that a corner of the original table is reached the first time the ball on the paste-up table comes to a place on the upper or lower edge where the corners of two copies of the original table are pasted together. For the 3 × 4 table this occurs at the first place where a multiple of the width equals a multiple of the length, namely, at 12 spaces along the edge. But the first places where these multiples can coincide is at the LCM of 3 and 4. In this case their LCM is their product because they have a GCF of 1. Remember that the LCM of two numbers a and b is just their product ab divided by their GCF, that is:

$$\text{LCM} = ab/\text{GCF}.$$

So, if l and w are the dimensions of the original rectangle, and GCF is their greatest common factor, then a new rectangle of large squares has dimension l/GCF and w/GCF. The ball hits a corner after hitting

$$(l/\text{GCF}) \times (w/\text{GCF}) \text{ large squares, or}$$

$$(l/\text{GCF}) \times (w/\text{GCF}) \times \text{GCF} = lw/\text{GCF} \text{ tiles.}$$

But lw/GCF = LCM. We now know for sure that the number of tiles hit by the ball is the least common multiple of the length and width of the table.

Now let's turn to the question of the number of ricochets of the ball. See if you can use the paste-up table to see that the number of ricochets is:

$$[(l + w)/\text{GCF}] - 2$$

The formula becomes the simpler one:

$$(l + w)/\text{GCF}$$

if you count both the starting and ending corners as a ricochet.

Relationships to the Ohio Academic Content Standards, 2002:

Grades 5-7:

Number, Number Sense and Operations Standard
The student will be able to…

- Apply and explain the use of prime factorizations, common factors, and common multiples in problem situations.

Patterns, Functions and Algebra Standard
The student will be able to…

- Describe, extend and determine the rule for patterns and relationships occurring in numeric patterns, computation, geometry, graphs and other applications.
- Use rules and variables to describe patterns, functions and other relationships.

Mathematical Processes Standard
The student will be able to…

- Apply and adapt problem-solving strategies to solve a variety of problems, including unfamiliar and non-routine problem situations.
- Relate mathematical ideas to one another and to other content areas; e.g., use area models for adding fractions, interpret graphs in reading, science and social studies.

Grades 8-10:

Number, Number Sense and Operations Standard
The student will be able to…

- Connect physical, verbal and symbolic representations of integers, rational numbers and irrational numbers.

Patterns, Functions and Algebra Standard
The student will be able to…

- Use algebraic representations, such as tables, graphs, expressions, functions and inequalities, to model and solve problem situations.

Mathematical Processes Standard
The student will be able to…

- Formulate a problem or mathematical model in response to a specific need or situation, determine information required to solve the problem, choose a method for obtaining this information, and set limits for acceptable solution.
- Apply mathematical knowledge and skills routinely in other content areas and practical situations.

Grades 11-12:

Mathematical Processes Standard
The student will be able to…

- Construct algorithms for multi-step and non-routine problems.

Relationships to the NCTM Principles and Standards, 2000:

Grades 6-8 and 9-12:

Number and Operations Standard
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Understand numbers, ways of representing numbers, relationships among numbers, and number systems.
- Compute fluently and make reasonable estimates.

Algebra Standard
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Understand patterns, relations, and functions.
- Use mathematical models to represent and understand quantitative relationships.

Problem Solving Standard
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Solve problems that arise in mathematics and in other contexts.

Connections Standard
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Recognize and apply mathematics in contexts outside of mathematics.