The $3x + 1$ Problem

One-Page Overview

By Robert B. Brown, The Ohio State University

<u>*Topics:*</u>

Numbers, Patterns

<u>*Level:*</u>

Grades 5 – 8

<u>*Problem:*</u>

Start with any whole number. Call it A. Find the next integer B in the following way:

If A is odd then $B = 3A + 1$, whereas if A is even, then $B = A/2$.

Continue generating numbers one after the other in this way.

Examples: $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$

$3 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

Do you always get to 1 regardless of which whole number you start with?

<u>*Getting Started:*</u>

Let the students try starting with different whole numbers. They will see that starting with 27 produces a surprising route. Mention that once you get to 1 you can stop, because after that the numbers cycle repeatedly 4, 2, 1, 4, 2, 1.

Ohio Academic Content Standards, 2002

5-7		8-10		11-12	
1. Number, Number Sense and Operations	X	1. Number, Number Sense and Operations	x	1. Number, Number Sense and Operations	
2. Measurement		2. Measurement		2. Measurement	
3. Geometry and Spatial Sense		3. Geometry and Spatial Sense		3. Geometry and Spatial Sense	
4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	X	4. Patterns, Functions and Algebra	
5. Data Analysis and Probability		5. Data Analysis and Probability		5. Data Analysis and Probability	
Mathematical Processes		**Mathematical Processes**		**Mathematical Processes**	
Reasoning		Reasoning			

NCTM Principles and Standards, 2000

6-8		9-12	
1. Number and Operations	X	1. Number and Operations	
2. Algebra	X	2. Algebra	
3. Geometry		3. Geometry	
4. Measurement		4. Measurement	
5. Data Analysis and Probability		5. Data Analysis and Probability	
6. Problem Solving		6. Problem Solving	
7. Reasoning and Proof	x	7. Reasoning and Proof	
8. Communication		8. Communication	
9. Connections		9. Connections	
10. Representation		10. Representation	

<u>Note</u>: Capital X denotes major emphasis; lower case x denotes minor emphasis.

Ohio's Systemic Initiative
Discovery

The *3x + 1* Problem

By Robert B. Brown, The Ohio State University

Topics: Numbers, Patterns	
Levels: Grades 5 – 8	*Timing:* 1-3 class periods
Materials: Pencil and paper; Blackboard or overhead	*Prerequisites:* None

Problem:

Start with any whole number. Call it *A*. Find the next integer *B* in the following way. If *A* is odd then $B = 3A + 1$, whereas if *A* is even, then $B = A/2$. Continue generating numbers one after the other in this way.

Examples: $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$

$3 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

Do you always get to 1 regardless of which whole number you start with?

Goals:

- Practice whole number calculations
- Look for patterns
- Practice describing patterns
- See how base 2 notation can help with this problem
- Practice cooperative group work, as this problem lends itself to letting each student work with different starting numbers
- Practice programming a computer or hand-held calculator

Big Ideas:

- A use for base 2 notation
- Tabulate everyone's results together
- No one has settled once and for all whether calculations starting with every positive integer eventually end up at 1.

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Procedure:

1. Pose the Problem. Let the students try starting with different whole numbers. They will see that starting with 27 produces a surprising route. Mention that once you get to 1 you can stop, because after that the numbers cycle repeatedly 4, 2, 1, 4, 2, 1....

2. One interesting question that comes up right away when you are looking for patterns is how long does it take before a number that you calculate is smaller than your initial number. For numbers like 5, 9, 13, 17, 21, ..., which are all of the form $4n + 1$, the third number that you calculate is always smaller than the number that you start with. This is because the sequence below will occur:

   $4n + 1$

   $3(4n + 1) + 1 = 12n + 4$

   $6n + 2$

   $3n + 1$, which is smaller than $4n + 1$.

4. What do 27, 31, 41, 47, 55, 63 have in common? What is the first number past 27 that is as interesting as 27?

   If x is odd, are there any similarities between the sequence starting at x and the sequence starting at $4x + 1$?

5. What can you say about the *average* behavior for numbers in some interval $a \leq x \leq b$? What if the length of the interval gets longer and longer? What if the length of the interval stays the same but the starting point of the interval gets larger and larger?

The Mathematics

This is a great activity because it is accessible to everyone from very young pupils and up. There are many patterns and partial patterns to be found by seeing what happens with different starting numbers. However, this problem has turned into one of the most tantalizing mathematical puzzles of the Twentieth Century, because no one has found a way to settle once and for all whether calculations starting with *every* positive integer eventually end up at 1. In fact, the problem seems so far from being settled that most research mathematicians will not even work on it. You don't want to tell that to your students right away, however, because if they know it is an unsolved problem, they probably won't delve into the investigation long enough to discover some of the nice patterns waiting for them.

Quite a bit can be said statistically about the problem, and because it is so easy to program, lots of data can be gathered by computer. Of course, a computer can also do at lightning speed the calculations for extremely large starting numbers.

Extensions:

If you write numbers in base 2 instead of base 10, then even numbers are distinguished by a 0 in the unit's place, and you divide by 2 by erasing the final 0. Odd numbers x have a 1 in the unit's digit, and you calculate $3x + 1$ in base 2 by shifting all the digits of x one position left and afterwards putting 1 in the unit's place. Then add the result to the original x in base 2 and you end up with $3x + 1$. Therefore, in base 2, the calculation of the successive numbers looks like manipulation of 0's and 1's. It is tantalizing to look for patterns within the 0's and 1's to see how they affect the calculations.

In base 2 the numbers of the form $4n + 1$ are those that end ...01. This makes the numbers of the form $4n + 3$ perhaps more interesting. They are the numbers that in base 2 end in ...11. In base 10 they are 3, 7, 11, 15, 19, 23, 27, 31, 35.... If students have done the calculations, they already know that 27 is interesting. What is the relationship between 27 and 31 in this problem?

Another extension is to start with negative integers instead of positive integers. You will readily find a cycle: -5, -14, -7, -20, -10, -5

Another variation is to replace the calculation $3x + 1$ by $ax + b$ for different selections of a and b. However, the most interesting situation remains $3x + 1$.

Reference:

Lagarias, Jeffrey C. (1985, Jan.). The $3x + 1$ problem and its generalizations. *American Mathematical Monthly, 92*(1), 3-23.

Ohio's Systemic Initiative Discovery

Relationships to the Ohio Academic Content Standards, 2002:

Grades 5-7:

Number, Number Sense and Operations Standard
The student will be able to…

- Apply number system properties when performing computations.

Patterns, Functions and Algebra Standard
The student will be able to…

- Describe, extend and determine the rule for patterns and relationships occurring in numeric patterns, computation, geometry, graphs and other applications.
- Use rules and variables to describe patterns, functions and other relationships.

Mathematical Processes Standard
The student will be able to…

- Use inductive thinking to generalize a pattern of observations for particular cases, make conjectures, and provide supporting arguments for conjectures.

Grades 8-10:

Patterns, Functions and Algebra Standard
The student will be able to…

- Generalize and explain patterns and sequences in order to find the next term and the *n*th term

Mathematical Processes Standard
The student will be able to…

- Apply reasoning processes and skills to construct logical verifications or counter-examples to test conjectures and to justify and defend algorithms and solutions.

Relationships to the NCTM Principles and Standards, 2000:

Grades 6-8:

Number and Operations Standard
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Understand numbers, ways of representing numbers, relationships among numbers and number systems.
- Compute fluently and make reasonable estimates.

Algebra Standard
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Understand patterns, relations, and functions.
- Use mathematical models to represent and understand quantitative relationships.

Reasoning and Proof Standard
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Make and investigate mathematical conjectures.