Ohio's Systemic Initiative Discovery

Volumes Mathematically

One-Page Overview

By Robert B. Brown, The Ohio State University

Topics:

Geometry and spatial sense, Problem-solving strategies

Levels:

Grades 7 – 12

Problem:

How does one go about finding the volume of an irregular shape?

Getting Started:

Break the class up into groups and ask them how they might go about finding out the volume of a square pyramid if they didn't already know it and if they couldn't look up the formula.

Ohio Academic Content Standards, 2002

5-7		8-10		11-12	
1. Number, Number Sense and Operations		1. Number, Number Sense and Operations		1. Number, Number Sense and Operations	
2. Measurement	X	2. Measurement	X	2. Measurement	X
3. Geometry and Spatial Sense	X	3. Geometry and Spatial Sense	X	3. Geometry and Spatial Sense	x
4. Patterns, Functions and Algebra		4. Patterns, Functions and Algebra		4. Patterns, Functions and Algebra	
5. Data Analysis and Probability		5. Data Analysis and Probability		5. Data Analysis and Probability	
Mathematical Processes Problem Solving, Representation		**Mathematical Processes** Problem Solving, Representation		**Mathematical Processes** Problem Solving, Representation	

NCTM Principles and Standards, 2000

6-8		9-12	
1. Number and Operations		1. Number and Operations	
2. Algebra		2. Algebra	
3. Geometry	X	3. Geometry	X
4. Measurement	X	4. Measurement	X
5. Data Analysis and Probability		5. Data Analysis and Probability	
6. Problem Solving	X	6. Problem Solving	X
7. Reasoning and Proof		7. Reasoning and Proof	
8. Communication		8. Communication	
9. Connections		9. Connections	
10. Representation	x	10. Representation	x

Note: Capital X denotes major emphasis; lower case x denotes minor emphasis.

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<table>
<tr><td colspan="2"><u>Topics:</u>
Geometry and spatial sense, Problem-solving strategies</td></tr>
<tr><td><u>Levels:</u>
Grades 7 – 12</td><td><u>Timing:</u>
Two one-hour periods</td></tr>
<tr><td><u>Materials:</u>
Cardboard, scissors, glue or tape</td><td><u>Prerequisites:</u>
Familiarity with simple solid figures such as cones, cylinders, pyramids, cubes , and spheres, as well as their radii, heights, and areas of their bases.
Experience with algebraic formulas.
Experience using the formula for the sum of squares of consecutive whole numbers: $1^2 + 2^2 + 3^2 + ... + n^2 = n(n + 1)(2n + 1)/6$
Pythagorean theorem (used for the volume of a sphere)</td></tr>
</table>

Problem:

How does one go about finding the volume of an irregular shape?

Goals:

- Experience an important nontrivial problem
- Deal with an intuitive limit
- Deal with the intuitive ideas underlying integration

Procedure:

1. Break the class into groups.
2. Ask the groups to brainstorm for 15 minutes on how they might find the volume of a square pyramid if they didn't already know it and they couldn't look up the formula.
3. Reconvene the whole class and have the groups report their ideas. Among the ideas that the students may come up with are these two. The first is to try to fit several

identical pyramids together to form a brick shape that has volume $l \times w \times h$. If that were possible, then the volume of the pyramid would be the volume of the brick divided by the number of pyramids used to form it. The other idea is to try to slice up the pyramid into thin slices and add together the volumes of all of the individual slices. Both ideas are developed in *The Mathematics* section below.

Extensions:

Try the same techniques to find the volume of a circular cone. The cone has circular slices, rather than the square slices in the pyramid, so in the appropriate spot in the calculation the area of a circle has to be used instead of the area of a square.

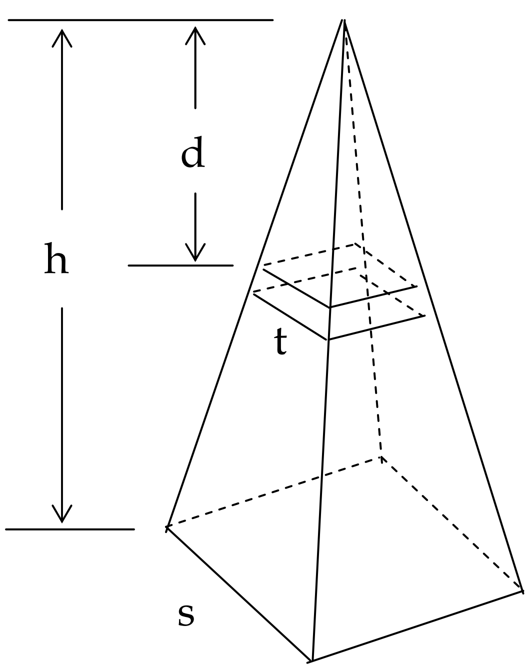
Try the same techniques to find the volume of a hemisphere. Double what you get and you have the volume of a sphere.

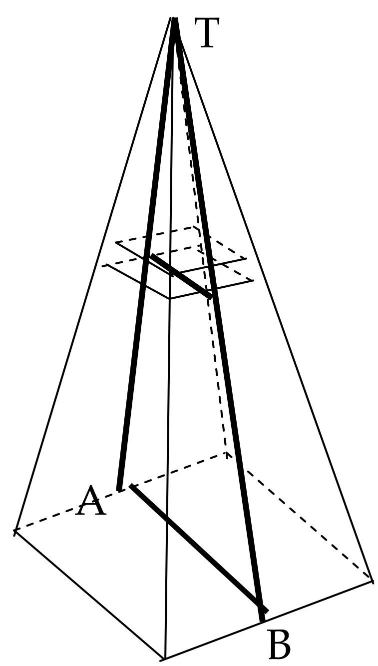
Closure:

The problem of finding the volumes of irregular shapes, like the problem of finding the areas of irregular figures, occupied scientists for centuries. Eventually the problem led to the invention of calculus by Newton and Leibnitz in the 17th Century, one of the epochal events in the history of mathematics and science. The technique of slicing is the basis of integration, one of the great ideas in calculus.

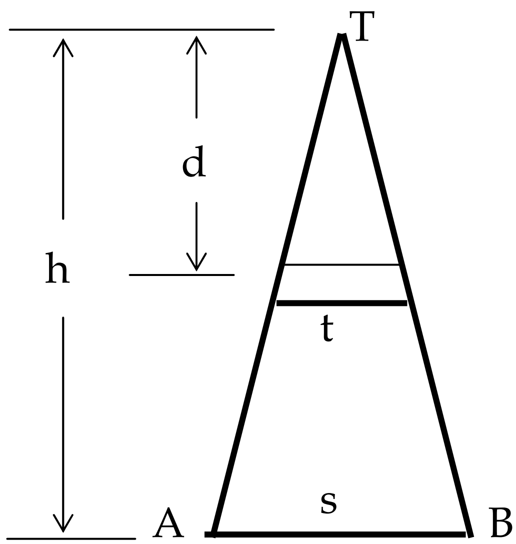
The Mathematics:

Here is how the volume of a square pyramid might be found by slicing it up into thin slices. Below on the left is a square pyramid with a thin slice drawn part way down the pyramid. Now cut the pyramid from top to bottom along the heavy lines shown in the right-hand drawing.





The section of the pyramid exposed by the cut looks like this:



You see two similar triangles. One is the whole section, and it has height h and base s. The other is the part of the section from the top down to where the slice ends. It is a triangle with height d and base t. These two triangles are similar. Because all ratios of corresponding lengths are the same in a pair of similar triangles, you know:

(*) $t/d = s/h$, and $t = d(s/h)$.

Now stick the pyramid together so it is whole again, but now run it through the slicer at the delicatessen and it comes out as a stack of very thin slices all of the same thickness. If n is the number of slices that come out, then the thickness of each one is h/n. If t is the side of a slice, then its volume is the area of the square t^2 times its thickness, that is, $t^2(h/n)$, which by (*) equals $d^2(s^2/h^2)(h/n)$ or:

(**) $d^2[s^2/(hn)]$.

This is actually just a little bit too much, because d measures the bottom of each slice. If instead d measured the top of each slice, then the volume $d^2[s^2/(hn)]$ would be a bit too small. The true volume is somewhere in between, but as we shall soon see, the difference is really, really small and we have a way to justify ignoring it altogether.

Looking at the volume (**), you see that the only term that changes from each slice to the next is d^2. The rest of the terms in the expression are constants for the pyramid. For the first top slice $d = h/n$, its thickness. For the second slice, $d = 2(h/n)$. For the third slice, $d = 3(h/n)$, and so on. We square each d and add up the volumes (**) to get the volume of the pyramid (remember that the result is just a little bit too big) to be (just a bit less than):

$$(h^2/n^2)[s^2/(hn)] + 2^2(h^2/n^2)[s^2/(hn)] + 3^2(h^2/n^2)[s^2/(hn)] + \ldots + n^2(h^2/n^2)[s^2/(hn)].$$

Ohio's Systemic Initiative Discovery

Think of the first term has having the hidden factor of 1, which is the same as 1^2, and factor out of each term the factor $(h^2/n^2)[s^2/(hn)]$, and you get for the volume:

$$(1^2 + 2^2 + 3^2 + ... + n^2)(h^2/n^2)[s^2/(hn)].$$

You can add this up explicitly because you already know the formula for the sum of consecutive squares, which is:

$$1^2 + 2^2 + 3^2 + ... + n^2 = n(n + 1)(2n + 1)/6.$$

So the volume (still a bit big) becomes:

$$\{n(n + 1)(2n + 1)/6\}(h^2/n^2)[s^2/(hn)].$$

This simplifies to:

$$\{n(n + 1)(2n + 1)/6\}(hs^2/n^3).$$

Now take the three factors of n in the denominator and divide one of them into each one of the three left-most factors in the expression to get:

$$\{[n/n][(n + 1)/n][(2n + 1)/n]\}(1/6)(hs^2) = \{[1][1 + 1/n][2 + 1/n]\}(1/6)(hs^2).$$

As you take thinner and thinner slices, the two little terms $1/n$ get closer and closer to zero, and the volume appears to become exactly:

$$\{[1][1][2]\}(1/6)(hs^2) = (1/3)(hs^2)$$

(which is the formula that you knew you would end up with all along).

But you would be justified in suspecting that it might be a little bit too big. Here is how to see that it isn't. Instead of measuring d to the bottom of each slice, measure it just to the top of each slice. Then the volume that you get will be a bit too small. The volume that you will get for the first slice is 0, which with some effort we can write as $0^2(h^2/n^2)[s^2/(hn)]$. The volume calculated for the second slice is now $1^2(h^2/n^2)[s^2/(hn)]$, for the third slice $2^2(h^2/n^2)[s^2/(hn)]$, and for the nth and last slice $(n - 1)^2(h^2/n^2)[s^2/(hn)]$. Now the sum of squares is

$$0^2 + 1^2 + 2^2 + ... + (n - 1)^2 = (n - 1)(n)(2n - 1)/6.$$

The volume (still a bit small) becomes:

$$\{(n - 1)(n)(2n - 1)/6\}(h^2/n^2)[s^2/(hn)].$$

This simplifies to:

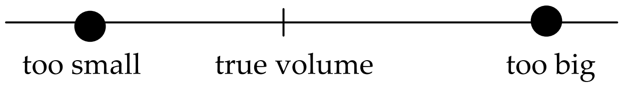
$$\{n(n + 1)(2n + 1)/6\}(hs^2/n^3).$$

Now take the three factors of n in the denominator and divide one of them into each one of the three left-most factors in the expression to get:

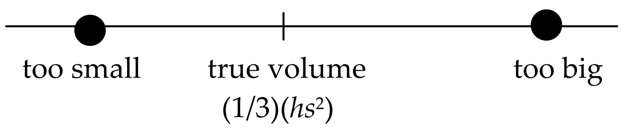
$$\{[(n - 1)/n][n/n][(2n - 1)/n]\}(1/6)(hs^2) = \{[1 - 1/n][1][2 - 1/n]\}(1/6)(hs^2).$$

Again the two little terms $1/n$ get closer and closer to zero, and the volume appears to become exactly $(1/3)(hs^2)$.

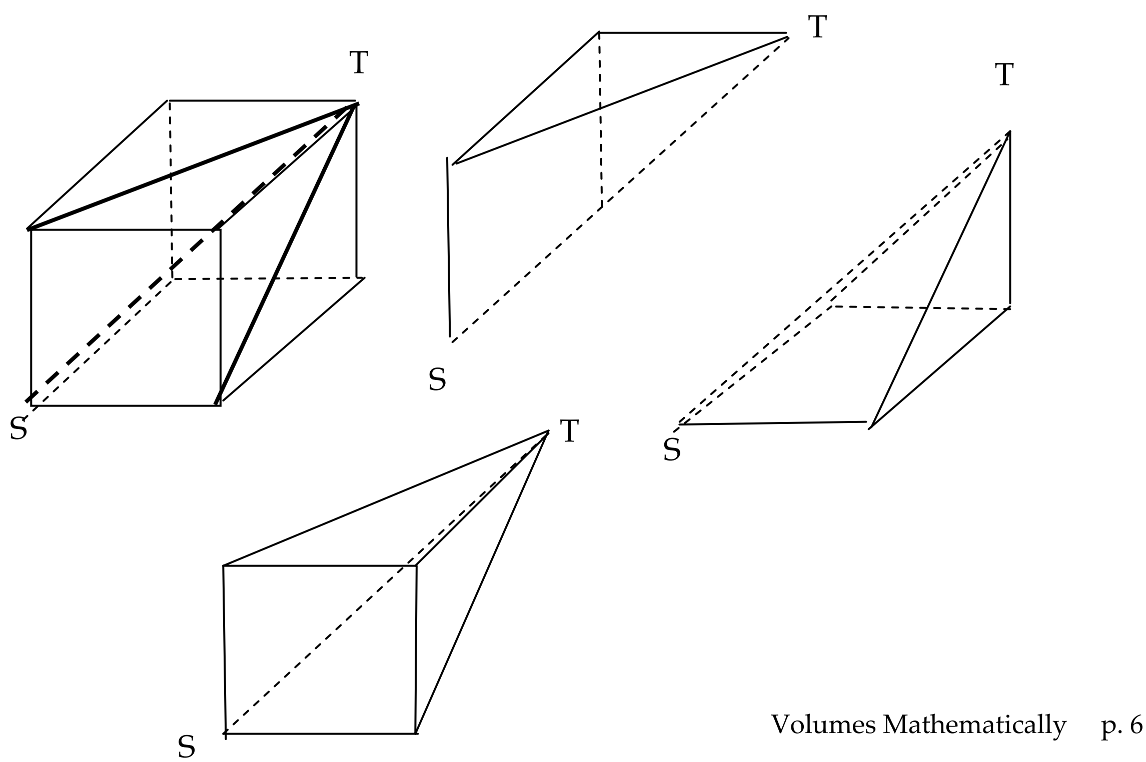
Now the situation is this. You have calculated a volume too small and one too big, with the true volume in between.



Now you let the slices get thinner and thinner. That puts a vise around “too small” and “too big” and as you turn the handle on the vise, “too small” and “too big” are squeezed right to the spot $(1/3)(hs^2)$. So the “true volume” must have been right at the same spot $(1/3)(hs^2)$ all the while because “true volume” is always trapped between the two “too’s”, no matter how tightly you clamp the vise.



The other way to “see” the volume of at least some pyramids is to see that a cube can be dissected into three congruent pyramids. Each pyramid is not nice and symmetrical with its top above the center of its base. Instead the top stands exactly above one of the corners of the base. Here is a drawing of how that can be done. It could be more easily seen if the students were to make three cardboard models of the pyramid and assemble them to get the cube. When assembled the tops of all three pyramids come together in the cube at the vertex T, and the long edge in each pyramid runs along the long diagonal ST of the cube.



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Discovery

Relationships to the Ohio Academic Content Standards, 2002:

Grades 8-10:

Measurement Standard
The student will be able to…

- Use formulas to find surface area and volume for specified three-dimensional objects accurate to a specified level of precision.
- Estimate and compute various attributes, including length, angle measure, area, surface area and volume, to a specified level of precision.

Geometry and Spatial Sense Standard
The student will be able to…

- Formally define geometric figures.
- Draw and construct representations of two- and three-dimensional geometric objects using a variety of tools, such as straightedge, compass and technology.

Mathematical Processes Standard
The student will be able to…

- Formulate a problem or mathematical model in response to a specific need or situation, determine information required to solve the problem, choose a method for obtaining this information, and set limits for acceptable solution.
- Use a variety of mathematical representations flexibly and appropriately to organize, record and communicate mathematical ideas.

Grades 11-12:

Measurement Standard
The student will be able to…

- Estimate and compute areas and volume in increasingly complex problem situations.

Mathematical Processes Standard
The student will be able to…

- Use formal mathematical language and notation to represent ideas, to demonstrate relationships within and among representation systems, and to formulate generalizations.

<u>Relationships to the NCTM Principles and Standards, 2000:</u>

<u>Grades 6-8 and Grades 9-12:</u>

<u>Geometry Standard</u>
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.

<u>Measurement Standard</u>
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Understand measurable attributes of objects and the units, systems, and processes of measurement.
- Apply appropriate techniques, tools, and formulas to determine measurements.

<u>Problem Solving Standard</u>
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Build new mathematical knowledge through problem solving.
- Apply and adapt a variety of appropriate strategies to solve problems.

<u>Representation Standard</u>
Instructional programs from pre-kindergarten through grade 12 should enable all students to…

- Create and use representations to organize, record, and communicate mathematical ideas.