Slides from Logarithms (Tutorial 21)

Logarithms

Some Important Ideas

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This handout contains selected slides to use when reviewing this tutorial topic with or without the video. To access all slides, open thumbnail link on the tutorial interface.

Goals

- **Common logarithms (base 10) and natural logarithms (base *e*) are used in business and in science.**
- **Logarithms are used to solve problems where values are not integers.**
- **Use a calculator to evaluate logarithms.**

Definition of Logarithm

A logarithmic function $y = \log_a(x)$ is the inverse of an exponential function of the form $f(x) = a^x$.

For $a > 0$ and $a \neq 1$, the logarithmic function with base a is written $f(x) = \log_a(x)$ with $\log_a(x) = y$ if and only if $a^y = x$.

Two Forms for Logarithms

- Common logarithms are written as $\log_{10}(x)$ or simply $\log(x)$ indicating the log function with base 10. From log definition, $a = 10$ and $y = \log_{10}(x)$ if and only if $10^y = x$.
- Natural logarithms are written as $\ln(x)$ or $\log_e(x)$ both forms indicating the log function with base *e*. From log definition, $a = e$ and $y = \ln(x)$ if and only if $e^y = x$.

Two Forms for Logarithms

Note:
- Like π, *e* is an irrational number.
- An *irrational number* cannot be written as the root of an algebraic equation with rational coefficients. To eight decimal places, *e* is approximately equal to (≈) 2.71828183.

Finding Logarithms Using a Calculator

✓ Evaluate $x = \log(8)$.

Recall $x = \log_{10}(8)$.

10 raised to what number *x* is equal to 8?

Using a calculator,

$$x = \log(8)$$
$$\approx .90308997$$

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Finding Logarithms Using a Calculator

✓ Evaluate $x = \ln(3)$.

Recall $x = \log_e(3)$.

e raised to what number x is equal to 3?

Using a calculator,

$x = \ln(3)$

≈ 1.098612289

Log and Exponent Relationship

Recall that a logarithm is the exponent of an exponential function.

- If $\log_a(b) = c$, then $b = a^c$ — Using the definition of a logarithm
- If $\log_{10}(100) = 2$, then $100 = 10^2$
  Evaluate log(100) with a calculator
- If $\log_5(125) = 3$, then $125 = 5^3$
  Cannot evaluate $\log_5(125)$ with a calculator
- If $\ln(e^4) = 4$, then $e^4 = e^4$.
  Evaluate $\ln(e^4)$ with a calculator

Solving Exponential Equations

Example 1: Given $14 = 2^x$. Find x.

Rewrite as a logarithmic equation

$\log_2(14) = x$

So far we know that x is between 3 and 4 because $2^3 = 8$ and $2^4 = 16$.

Solving Exponential Equations

Change of Base Rule: $\log_a(b) = \frac{\log_d(b)}{\log_d(a)}$

Example 2:
Solve the exponential equation $71 = 8^x$.

Rewrite as a logarithmic equation

$\log_8(71) = x$

Apply change of base rule

$x = \log_8(71) = \frac{\log(71)}{\log(8)}$ base 10

or

$x = \log_8(71) = \frac{\ln(71)}{\ln(8)}$ base e

Use a calculator to evaluate either of the above logarithmic equations and find x. $x \approx 2.049915707$

Solving Exponential Equations

Check that $x \approx 2.049915707$ is a solution to $71 = 8^x$

Using a calculator:

1. Raise 8 to 2.049915707. The result will not be exactly 71 because of calculator truncation.
2. Verify the exact solution by storing the value $\frac{\ln(71)}{\ln(8)}$ in calculator as x.
   Raise 8 to the stored value.
   The calculator answer is now 71 exactly.

Transformations With Log or Ln

1. Power Rule: $\log_a(u^n) = n \cdot \log_a(u)$

   Inverse Properties: If $a > 0$ and $a \neq 1$, then for any real number u

2. $a^{\log_a(u)} = u$

3. $\log_a(a^u) = u$ for $u>0$

- Since $a^0 = 1$ and $e^0 = 1$, $\log_a(1) = 0$ and $\ln(1) = 0$.
- Since $e^1 = e$, $\ln(e) = 1$ or more generally $\log_a(a) = 1$.

Note: Taking the common logarithm, natural logarithm, or inverse logarithm (antilogarithm) of each side of an equation is a valid operation.

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Transformations With Log or Ln

Example 1: Given $14 = 2^x$. Solve for x.

Take the log of each side of the equation

$\ln(14) = \ln(2^x)$

Apply the power rule to 2^x

$\ln(14) = x \cdot \ln(2)$

Solve for x by evaluating the logarithms

$\frac{\ln(14)}{\ln(2)} = x$

$3.807354922 \approx x$

What is *e*?

- e is defined as $\lim(1 + \frac{1}{n})^n$ as n goes to infinity.

Try letting n=2, n=100, or any large value and evaluate with your calculator.

- e is an irrational number
- e is approximately equal (≈) to 2.718281828459 or $e \approx 2.718281828459$

Super Red Tomato Juice

The pH is a measure of a solution's acidity or alkalinity. A low pH indicates an acidic solution and a high pH indicates a base solution. The pH of a solution is related to the concentration of hydrogen ions by the formula $pH = \log(\frac{1}{H^+})$, where H^+ is the number of ions per liter.

Super Red Tomato Juice

If the pH of super red tomato juice is 3.9, what is the concentration of hydrogen ions?

Using the formula $pH = \log(\frac{1}{H^+})$

Substitute in formula $3.9 = \log(\frac{1}{H^+})$

Take the inverse log (antilog) of each side of the equation

$\text{antilog}(3.9) = \text{antilog}(\log(\frac{1}{H^+}))$

Note: Log and log inverse (antilog) are inverse functions and $\log^{-1}(x) = 10^x$

Super Red Tomato Juice

Simplify $\text{antilog}(3.9) = \text{antilog}(\log(\frac{1}{H^+}))$

$\text{antilog}(3.9) = \frac{1}{H^+}$

Solve for H^+

$H^+ = \frac{1}{\text{antilog}(3.9)}$

Evaluate the antilog with the 2nd log key

$H^+ = \frac{1}{7943.282347}$

Simplify $H^+ \approx 1.258925412E-4$

The concentration of hydrogen ions in super red tomato juice is approximately 1.26×10^{-4} hydrogen ions per liter of solution.

Newton's Law of Cooling

Newton's Law of Cooling formula $T(t) = T_a + (T_i - T_a)e^{-kt}$

$T(t)$ is temperature of soup at time t

T_a = 70° F ambient temperature (room)

T_i = 184° F initial temperature of the cooling object (soup)

k = 0.05 is a constant related to the cooling object

$a = T_i - T_a$ = 184°F − 70°F = 114°F

t = time in minutes

Soup is heated to 184°F in a room with a temperature of 70°F.

How long will it take for the soup to cool to 90°F and be ready to eat?

Solve $T(t) = T_a + (T_i - T_a)e^{-kt}$ *for t.*

Substitute values in formula

$90°F = 70°F + (184°F - 70°F)e^{-0.05t}$

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Newton's Law of Cooling

How long will it take for the soup to cool to 90°F?

Solve $T(t) = T_a + (T_i - T_a)e^{-kt}$ for t.

Substitute values in formula

$$90°F = 70°F + (184°F - 70°F)e^{-0.05t}$$

Combine terms in parentheses

$$90°F = 70°F + (114°F)e^{-0.05t}$$

Subtract 70°F from both sides and divide both sides by 114°F

$$20°F = (114°F)e^{-0.05t}$$

$$\left(\frac{20°F}{114°F}\right) = e^{-0.05t}$$

Take the natural logarithm (ln) of each side

$$\ln\left(\frac{20}{114}\right) = \ln(e^{-0.05t})$$

Newton's Law of Cooling

How long will it take for the soup to cool to 90°F?

Take the ln of each side $\ln\left(\frac{20}{114}\right) = \ln(e^{-0.05t})$

Apply power rule $\ln\left(\frac{20}{114}\right) = -0.05t \cdot \ln(e)$

Note: ln (*e*) = 1 $\ln\left(\frac{20}{114}\right) = -0.05t$

Evaluate ln with a calculator and divide both sides by -0.05

$-1.740466175 \approx -0.05\,t$

$34.8093235 \approx t$

The soup will be ready to eat in approximately 34.8 minutes!

Compound Interest

Present value formula for compound interest

$$P = A\left(1 + \frac{r}{n}\right)^{nt}$$

- *A* is the amount of the initial deposit
- *P* is the final amount
- *r* is the rate as a decimal
- *n* is the number of times compounded per year
- *t* is the time in years

$10,000 is deposited in a bank at a rate of 5% per year compounded monthly.
How long should the money be in the bank to grow to $15,000?

Substitute values into formula

$$\$15{,}000 = \$10{,}000\left(\frac{1+.05}{12}\right)^{12t}$$

Compound Interest

Solve for t $\$15{,}000 = \$10{,}000(1 + .05)^{12t}$

Divide each side by $10,000 $\frac{\$15{,}000}{\$10{,}000} = \left(\frac{1+.05}{12}\right)^{12t}$

Simplify and combine terms

$$1.5 = (1.00416\overline{6})^{12t}$$

Take the common logarithm (log) of each side

$$\log(1.5) = \log(1.00416\overline{6})^{12t}$$

Apply the power rule

$$\log(1.5) = 12t \cdot \log(1.00416\overline{6})$$

Compound Interest

Solve for *t* $\$15{,}000 = \$10{,}000(1 + .05)^{12t}$

$$\log(1.5) = 12t \cdot \log(1.00416\overline{6})$$

Divide each side by log(1.004166) $\frac{\log(1.5)}{\log(1.00416\overline{6})} = 12t$

Evaluate the logs and divide each side by 12

$97.52979038 = 12t$

$8.126314585 \approx t$

It will take approximately 8 years for the money in the bank to grow to $15,000.

Summary

- Logarithms base 10 are called **common logarithms**
- Logarithms base e are called **natural logarithms**
- Logarithms are used to solve problems in chemistry, physics, and business.